Electromagnetic Theory I

Examination

Instructions: This is a closed book exam: no notes or devices allowed except your mobile phone for the sole purpose of uploading your solutions to canvas. The only other resource allowed is the formula sheet provided as part of the exam. Answer the following N questions, each of which is worth 100/N points. Explain your reasoning in each case. Use SI units throughout. Points will be deducted for gaps in logic as well as for errors in calculation. *Extra points will be deducted if an incorrect answer is presented with incorrect units!* A sheet of useful formulas is provided in the last 2 pages.

Formulae

Maxwell's Equations (SI units)

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_{free} \qquad \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \mathbf{J}_{free} + \frac{\partial \mathbf{D}}{\partial t} \\ \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \qquad \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \end{aligned}$$

Multipole Moments

$$p^k = \int d^3r r^k \rho(\mathbf{r}) \qquad Q^{km} = \int d^3r (3r^k r^m - \delta_{km}\mathbf{r}^2)\rho(\mathbf{r}) \qquad m^k = \frac{1}{2}\int d^3r [\mathbf{r} \times \mathbf{J}(\mathbf{r})]^k$$

Laplacian in Cylindrical Coordinates

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

Laplacian in Spherical Coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

Legendre polynomials and Spherical Harmonics

$$\begin{aligned} \text{Rodrigues}: P_l(x) &= \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \qquad \text{Orthogonality}: \int_{-1}^{1} dx P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'} \\ P_l^m &= \frac{(-)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \\ Y_{lm}(\theta,\varphi) &= \sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi} \end{aligned}$$

Energy and Momentum Densities (Linear materials: $\boldsymbol{D} = \epsilon \boldsymbol{E}$, $\boldsymbol{B} = \mu \boldsymbol{H}$)

$$u = \frac{1}{2}\left(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}\right), \qquad \boldsymbol{g} = \boldsymbol{D} \times \boldsymbol{B}$$

Energy Flux and Stress Tensor (Linear materials: $\boldsymbol{D} = \epsilon \boldsymbol{E}$, $\boldsymbol{B} = \mu \boldsymbol{H}$)

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}, \qquad T^{ij} = E^i D^j + H^i B^j - \delta_{ij} u$$

Covariant Form of Maxwell Equations

$$F_{\mu\nu} = c(\partial_\mu A_\nu - \partial_\nu A_\mu) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & cB^3 & -cB^2 \\ E_2 & -cB^3 & 0 & cB^1 \\ E_3 & cB^2 & -cB^1 & 0 \end{pmatrix}, \qquad J^\mu = (\rho c, \boldsymbol{J})$$

$$\epsilon_0 \partial_\rho F^{\mu\rho} = \frac{1}{c} J^\mu, \qquad \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0, \qquad \frac{dp_\mu}{d\tau} = \frac{q}{mc} F_{\mu\nu} p^\nu$$

Lorentz Transformation of EM Fields

$$\boldsymbol{E}'_\parallel = \boldsymbol{E}_\parallel, \qquad \boldsymbol{B}'_\parallel = \boldsymbol{B}_\parallel$$

$$\boldsymbol{E}'_\perp = \gamma(\boldsymbol{E}_\perp + \boldsymbol{v} \times \boldsymbol{B}), \qquad \boldsymbol{B}'_\perp = \gamma\left(\boldsymbol{B}_\perp - \frac{\boldsymbol{v}}{c^2} \times \boldsymbol{E}\right)$$

where the primed frame moves with velocity $\boldsymbol{v}$ relative to the unprimed frame, and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Coordinate Transformation under a Boost in the x-direction

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma(t - vx/c^2)$$

Green Functions

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G = \delta(t - t')\delta(\boldsymbol{r} - \boldsymbol{r}'), \quad G_r = \frac{\delta(t - t' - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|}$$

$$\left(-\nabla^2 - k^2\right) G_k = \delta(\boldsymbol{r} - \boldsymbol{r}'), \qquad G_k = \frac{e^{ik|\boldsymbol{r} - \boldsymbol{r}'|}}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|}$$

Spherical Bessel Functions

$$x \to 0: \qquad j_l(x) \sim \frac{x^l}{(2l+1)!!}, \qquad n_l(x) \sim -\frac{(2l-1)!!}{x^{l+1}}$$

$$x \to \infty: \qquad j_l \sim \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right), \qquad h_l^{(1)} \sim (-i)^{l+1}\frac{e^{ix}}{x}$$

$$j_0(x) = \frac{\sin x}{x}, \qquad n_0(x) = -\frac{\cos x}{x}, \qquad h_l^{(1,2)} = j_l \pm i n_l \tag{1}$$

Vector Spherical Harmonics

$$\boldsymbol{X}_{lm} \equiv \frac{1}{\sqrt{l(l+1)}} \boldsymbol{L} Y_{lm}, \qquad \int d\Omega\, \boldsymbol{X}^*_{l'm'} \cdot \boldsymbol{X}_{lm} = \delta_{ll'}\delta_{mm'}$$

$$\boldsymbol{\epsilon}_0 e^{ikz} = \frac{1}{2}\sum_{lm} i^l \sqrt{4\pi(2l+1)}\left[\frac{i}{k}\nabla \times j_l(kr) i(\epsilon_{0+}\boldsymbol{X}_{l-1} - \epsilon_{0-}\boldsymbol{X}_{l1}) + j_l(kr)(\epsilon_{0+}\boldsymbol{X}_{l-1} + \epsilon_{0-}\boldsymbol{X}_{l1})\right]$$

$$\boldsymbol{H}_0 = \frac{1}{2Z_0}\sum_{lm} i^l \sqrt{4\pi(2l+1)}\left[j_l(kr) i(\epsilon_{0+}\boldsymbol{X}_{l-1} - \epsilon_{0-}\boldsymbol{X}_{l1}) - \frac{i}{k}\nabla \times (j_l(kr)(\epsilon_{0+}\boldsymbol{X}_{l-1} + \epsilon_{0-}\boldsymbol{X}_{l1})\right]$$