String Theory

Problem Set 3

Due: 3 March 2008

1. The vertex operator for the emission of the "gluon" state in the bosonic string model, $\epsilon(k) \cdot a_{-1} e^{ik\cdot q_0}|0\rangle$, is
$$V(k,\epsilon) \equiv : \epsilon(k) \cdot \mathcal{P}(z) e^{ik\cdot x(z)} : \tag{1}$$
Using this expression calculate the 4 "gluon" scattering amplitude
$$\langle 0|\epsilon_4 \cdot a_1 e^{ik_4\cdot q_0} V(k_3,\epsilon_3) \frac{1}{L_0 - 1} V(k_2,\epsilon_2) \epsilon_1 \cdot a_{-1} e^{ik_1\cdot q_0}|0\rangle. \tag{2}$$
Compare your answer to the 4 gluon calculation in Yang-Mills theory that you did in problem 4 of set 2.

2. In class we have obtained the Virasoro algebra, including the subtle c-number term, using the mode expansion and a careful treatment of normal ordering. Another approach is to work in complex z space: $\mathcal{P}(z) \equiv \sum_n a_n z^{-n}$, and use point splitting to define products of field operators.
   (a) Assuming $|z'| < |z|$, express $\mathcal{P}(z)\mathcal{P}(z')$ as its normal ordered form plus a function $f(z,z')$, which you are to obtain. Then one can define $:\mathcal{P}^2(z):$ by subtracting this function from $\mathcal{P}(z)\mathcal{P}(z')$ and taking $z' \to z$.
   (b) Using the above definition of normal ordering analyze the product $:\mathcal{P}^2(z)::\mathcal{P}^2(y):$, for $|y| < |z|$, by normal ordering the complete product using the Wick expansion, writing it as $:\mathcal{P}^2(z)\mathcal{P}^2(y):$ plus a quadratic normal ordered term plus a c-number term. Obtain all the coefficient functions of this operator product expansion. Note that the more singular terms have simpler operator properties.
   (c) Recall that $L_n = \oint z^n : \mathcal{P}^2(z) : dz/(4\pi i z)$. Use (a) and (b) to calculate $[L_n, L_m]$ via contour deformation. Note that you will have to obtain the residues of double and quartic poles at $z' = z$!

3. Complete the evaluation of the Virasoro algebra we sketched in class for half integer (b_r) and integer (d_m) moded fermionic oscillators. Recall that you need to get the operator part of the right side for the half integer case and must do the complete calculation for the integer moded case. You may either work with the normal mode basis as in class or try your hand with the operator product approach developed in problem 2. Make for yourself a summary table of all cases, fermi, bose, integer and half integer. I list some of our notations and definitions below:
$$\{b_r, b_s\} = \delta_{rs}, \qquad \{d_l, d_m\} = \delta_{ml}$$
$$L_n^b = \frac{1}{2}\sum_{r=-\infty}^{\infty}\left(r + \frac{n}{2}\right) : b_{-r} b_{r+n} :, \qquad L_n^d = \frac{1}{2}\sum_{m=-\infty}^{\infty}\left(m + \frac{n}{2}\right) : d_{-m} d_{m+n} :$$