String Theory

Problem Set 5

Due: 8 April 2008

1. The Jacobi imaginary transformation can be proved using the Poisson resummation technique on the series definition of the θ functions. The technique expresses a sum like $\sum_n e^{-\alpha n^2+\beta n}$ as one of the form $\sum_n e^{-\pi^2 n^2/\alpha+i\pi\beta n/\alpha}$

   (a) Prove

   $$\sum_n e^{-\alpha n^2+\beta n} = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/4\alpha} \sum_m e^{-\pi^2 m^2/\alpha+i\pi\beta m/\alpha}. \tag{1}$$

   This procedue is to insert $1 = \int dy\delta(y-n)$ in the sum and then replace n in the summand by y. Next write $\delta(y-n) = \int d\lambda e^{i\lambda(y-n)}/2\pi$ and do the sum over n using $\sum_n e^{-i\lambda n} = 2\pi \sum_m \delta(\lambda - 2m\pi)$. (Remember that the sum on the left here is periodic under $\lambda \to \lambda + 2\pi$!) Now the y integral is just a gaussian that can be done by completing the square and the λ integral is trivial.

   (b) Use the identity you proved in a) to prove the Jacobi imaginary transformation:

   $$\begin{aligned} \theta_1(z,q) &\equiv -i \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} e^{(2n+1)iz}(-)^n \\ &= -i\sqrt{\frac{\pi}{-\ln q}} e^{z^2/\ln q} \theta_1(-i\pi z/\ln q, e^{\pi^2/\ln q}) \\ &= i\sqrt{\frac{\pi}{-\ln q}} e^{z^2/\ln q} \theta_1(i\pi z/\ln q, e^{\pi^2/\ln q}) \end{aligned} \tag{2}$$

2. Calculate the 4 bosonic string scattering amplitude from the opeartor formalism with two nonadjacent particles, say 3, 1, with Koba-Nielsen variables 0 and ∞ respectively. This means that one of the two vertex operators for the remaining particles will be $V =: e^{k_2\cdot(iq_0+\sqrt{2\alpha'}\sum_{n\neq 0} a_{-n}/n)}$: and the other is $V_T =: e^{k_4\cdot(iq_0+\sqrt{2\alpha'}\sum_{n\neq 0}(-)^n a_{-n}/n)}$: Show that you now have to add two diagrams to get the complete answer:

   $$\langle 0, k_1 | V(k_2) \frac{1}{L_0 - 1} V_T(k_4) | 0, k_3 \rangle + \langle 0, k_1 | V_T(k_4) \frac{1}{L_0 - 1} V(k_2) | 0, k_3 \rangle \tag{3}$$

   In the strip picture of the worldsheet, these matrix elements correspond to two particles being emitted or absorbed from opposite sides of the strip.

3. Repeat the previous problem for the 4 pion function in the Neveu-Schwarz model using picture 2. Take care with the minus signs!

4. The proof of the Ward identities corresponding to the G_r in the Neveu-Schwarz model must be reexamined when both V and V_T are present in the state representing, say, the right half of a tree. The cancelled propagator argument is not valid when the cancelled propagator occurs between a V and a V_T. But in this case different orderings of the V's with respect to the V_T's must be explicitly included. Show that cancelled propagator terms from different orderings then cancel each other. You may take the simplest case of one V and one V_T. Note that the different orderings must be taken with the correct signs (see the previous problem)!