Quantum Field Theory I

Problem Set 7

Due: 14 November 2007

1. S, Problem 48.4

2. Prove, without using perturbation theory, that the function

$$\frac{\langle out|\, T[\psi_\alpha(x)\bar{\psi}_\beta(y)]\, |in\rangle}{\langle out|in\rangle}$$

in the presence of an external electromagnetic field A_μ is a Green's function for the differential operator $[i\gamma\cdot\partial + q\gamma\cdot A - m]$. You will need the facts that the operator $\psi(x)$ obeys the Dirac equation in the external field $A(x)$, that $d\theta/dt = \delta(t)$ and that the anticommutation relations for $\psi(x), \overline{\psi}(y)$ at $x^0 = y^0$ are not changed by the presence of A.

3. Pair production in an external EM field

   a) Write down the lowest order contribution to the amplitude for producing an electron-positron pair in the presence of an external field A_μ. Which Fourier components of $A_\mu(x)$ contribute to pair production to lowest order in qA? What will happen to the vacuum persistence amplitude $\langle out|in\rangle$ if pair production is possible?

   b) Calculate the differential probability of producing a pair with momenta $\mathbf{q}_1$, $\mathbf{q}_2$ and unobserved helicities, in terms of $\tilde{A}_\mu$, the Fourier transform of $A_\mu(x)$. [You will need

$$\sum_\lambda u_\lambda(p)\overline{u}_\lambda(p) = m - \gamma\cdot p, \qquad \sum_\lambda v_\lambda(p)\overline{v}_\lambda(p) = -m - \gamma\cdot p$$

   and the formulae for the trace of products of γ matrices. Remember that $\tilde{A}$ is complex and that $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$.] For the special case $\mathbf{q}_1 = \mathbf{q}$, $\mathbf{q}_2 = -\mathbf{q}$, $q_1^0 = q_2^0 = \omega(\mathbf{q})$, verify that your answer is positive and that it does not depend on $\tilde{A}^0$, and express it in terms of Fourier components of the electromagnetic field $F_{\mu\nu}$. [This verifies the gauge invariance of the calculation.]

   c) The total probability for producing a pair is the integral of the result of part b) over $\mathbf{q}_1$, $\mathbf{q}_2$. This is a six dimensional integral but the external field depends only on the four components of $K = q_1 + q_2$. Integration over the remaining two variables can be carried out by the following procedure. Insert $1 = \int d^4K\,\delta(K - q_1 - q_2)$ into the integrand and replace the arguments of the fields by K. Then one needs the values of

$$F(K^2) \equiv \int \frac{d^3q_1}{2\omega_1}\frac{d^3q_2}{2\omega_2}\delta(K - q_1 - q_2)$$
$$G^{\mu\nu}(K) \equiv \int \frac{d^3q_1}{2\omega_1}\frac{d^3q_2}{2\omega_2}q_1^\mu q_2^\nu \delta(K - q_1 - q_2).$$

By considering G^μ_μ and $K_\mu K_\nu G^{\mu\nu}$ and Lorentz covariance, show that

$$G^{\mu\nu} = \left[\left(\frac{1}{6} - \frac{m^2}{3K^2}\right)K^\mu K^\nu + \left(\frac{K^2}{12} + \frac{m^2}{3}\right)\eta^{\mu\nu}\right]F(K^2).$$

Since $F(K^2)$ is a Lorentz invariant, it may be evaluated in any frame. By choosing the frame in which $\mathbf{K} = 0$, show that

$$F(K^2) = \frac{\pi}{2}\sqrt{1 + \frac{4m^2}{K^2}}\theta(K^0)\theta(-K^2 - 4m^2),$$

where the step functions simply reflect the fact that in the center of mass frame of the two particles, $q_1^0 + q_2^0 \geq 2m$.

d) Using the results of part c), express the total probability for the production of a pair with unobserved momenta and helicities as an integral over K of an explicit function of K.

4. S, Problem 49.1