Regularization of Landau Ghost

Let us first write the solution of the renormalization group equation in the form

$$\ln \frac{R_0}{R} = -\int_{\lambda(R)}^{\lambda(R_0)} \frac{dx}{\psi(x)}$$
$$-\psi(x) = b_1 x^2 + b_2 x^3 + \cdots \tag{1}$$

If the integral on the right converges as $x \to \infty$, $\lambda(R)$ blows up at a finite $R = R_\infty$ given by

$$\ln \frac{R_\infty}{R_0} = -\int_{\lambda(R_0)}^{\infty} \frac{dx}{\psi(x)}. \tag{2}$$

This is the Landau ghost problem. If $\psi(x)$ is approximated by the truncation $\psi_T(x)$ of its perturbation series at any finite order the integral of $1/\psi_T$ will converge and the Landau ghost will be present. To avoid the problem $\psi(x)$ must grow slower than linearly with x as $x \to \infty$.

In the 1970's Richardson (Phys. Lett. **82** (1979) 272) noted a very simple way to regulate this disease. He worked in momentum space wherein the one-loop RG improved Coulomb potential has the form $1/(\boldsymbol{q}^2 \ln(\boldsymbol{q}^2/\Lambda^2))$. Richardson simply added 1 to the argument of the logarithm, which not only removes the Landau ghost, but also arranges the infrared behavior $1/\boldsymbol{q}^4$ characteristic of a linear confining potential.

Writing the solution of the renormalization group equation using ψ_T in place of the exact ψ in the form

$$\ln \frac{R_\infty}{R} = -\int_{\lambda(R)}^{\infty} \frac{dx}{\psi_T(x)}$$
$$-\psi_T(x) = b_1 x^2 + b_2 x^3 + \cdots + b_T x^T \tag{3}$$

we generalize Richardson's idea by replacing the left side with

$$\ln \frac{R_\infty}{R} \to \frac{1}{p} \ln\left(1 + \frac{R_\infty^p}{R^p}\right). \tag{4}$$

We note that sending $p \to \infty$ with $R < R_\infty$ regains the original truncated solution. The large coupling behavior of the right side of the solution is

$$-\int_{\lambda(R)}^{\infty} \frac{dx}{\psi_T(x)} \sim \frac{1}{(T-1) b_T \lambda^{T-1}} \tag{5}$$

so we see that the regularized coupling has the large R behavior

$$\lambda(R) \sim \left[\frac{p R^p}{(T-1) b_T R_\infty^p}\right]^{1/(T-1)} \tag{6}$$

For $p = 2(T-1)$ this corresponds to an asymptotically linear confining static potential whereas for $p/(T-1) > 2$ the static potential is over-confining.

Finally we infer the new ψ function, $\psi_r(\lambda)$ corresponding to this Richardson regularization:

$$\psi_r(\lambda) \equiv -R\frac{d\lambda(R)}{dR} = \psi_T(\lambda)\left(1 - \exp\left\{p \int_{\lambda}^{\infty} \frac{dx}{\psi_T(x)}\right\}\right) \tag{7}$$

Recalling that $\psi_T < 0$ we see that ψ_r differs from ψ_T by terms that vanish to all orders in perturbation theory.