Phase space for N massless particles

$$
\begin{aligned}
P &= \int \prod_k \frac{dp_k^+ d^2p_k}{(2\pi)^3 2p_k^+}(2\pi)^4\delta\left(\sum_k p_k^+ - \frac{E}{\sqrt{2}}\right)\delta\left(\sum_k \boldsymbol{p}_k\right)\delta\left(\sum_k \frac{\boldsymbol{p}_k^2}{2p_k^+} - \frac{E}{\sqrt{2}}\right) &(1)\\
&= \int \prod_k \frac{dp_k^+ d^2p_k}{(2\pi)^3 2p_k^+}(2\pi)^4\delta\left(\sum_k p_k^+ - \frac{E}{\sqrt{2}}\right)\int \frac{dx^+ d^2x}{(2\pi)^3}\exp\left(i\boldsymbol{x}\cdot\sum_k \boldsymbol{p}_k - ix^+\left(\sum_k \frac{\boldsymbol{p}_k^2}{2p_k^+} - \frac{E}{\sqrt{2}}\right)\right)\\
&= \int \prod_k \frac{dp_k^+ d^2p_k}{(2\pi)^3 2p_k^+}(2\pi)^4\delta\left(\sum_k p_k^+ - \frac{E}{\sqrt{2}}\right)\int \frac{dx^+ d^2x}{(2\pi)^3}\exp\left(i\frac{E}{2\sqrt{2}x^+}\boldsymbol{x}^2 - ix^+\left(\sum_k \frac{\boldsymbol{p}_k^2}{2p_k^+} - \frac{E}{\sqrt{2}}\right)\right)\\
&= \int \prod_k \frac{dp_k^+ d^2p_k}{(2\pi)^3 2p_k^+}(2\pi)^4\delta\left(\sum_k p_k^+ - \frac{E}{\sqrt{2}}\right)\int \frac{dx^+}{(2\pi)^3}\frac{2\sqrt{2}i\pi x^+}{E}\exp\left(-ix^+\left(\sum_k \frac{\boldsymbol{p}_k^2}{2p_k^+} - \frac{E}{\sqrt{2}}\right)\right)\\
&= \int \prod_k \frac{dp_k^+}{(2\pi)^3}(2\pi)^4\delta\left(\sum_k p_k^+ - \frac{E}{\sqrt{2}}\right)\frac{1}{\pi E}\frac{\partial}{\partial E}\int \frac{d^2p_k}{2p_k^+}\delta\left(\sum_k \frac{\boldsymbol{p}_k^2}{2p_k^+} - \frac{E}{\sqrt{2}}\right)\\
&= \int \prod_k \frac{dp_k^+}{(2\pi)^3}(2\pi)^4\delta\left(\sum_k p_k^+ - \frac{E}{\sqrt{2}}\right)\frac{1}{\pi E}\frac{\partial}{\partial E}\int d^{2N}Q\ \delta\left(\boldsymbol{Q}^2 - \frac{E}{\sqrt{2}}\right)\\
&= \left(\frac{E}{\sqrt{2}}\right)^{N-1}\int \prod_k \frac{dx_k}{(2\pi)^3}(2\pi)^4\delta\left(\sum_k x_k - 1\right)\frac{1}{\pi E}\frac{\partial}{\partial E}\left(\frac{E}{\sqrt{2}}\right)^{N-1}\int d^{2N}Q\ \delta\left(\boldsymbol{Q}^2 - 1\right)\\
&= \frac{N-1}{2^{N-2}}\frac{E^{2N-4}}{(2\pi)^{3N-3}}\int_0^\infty \prod_k dx_k\delta\left(\sum_k x_k - 1\right)\int d^{2N}Q\ \delta\left(\boldsymbol{Q}^2 - 1\right) &(2)
\end{aligned}
$$

The integral over $\boldsymbol{Q}$ is just proportional to the volume of a $2N-1$ dimensional sphere:

$$
\int d^{2N}Q\ \delta\left(\boldsymbol{Q}^2 - 1\right) = \frac{\pi^N}{\Gamma(N)} \tag{3}
$$

and a short calculation shows that

$$
\int_0^\infty \prod_k dx_k\delta\left(\sum_k x_k - 1\right) = \frac{1}{\Gamma(N)}. \tag{4}
$$

The same method works for any **even** transverse dimension $d \equiv D-2$ where D is the spacetime dimension. Instead of (4), we need

$$
\int_0^\infty \prod_k dx_k x_k^{-1+d/2}\delta\left(\sum_k x_k - 1\right) = \frac{\Gamma(d/2)^N}{\Gamma(Nd/2)}. \tag{5}
$$

Then following identical steps in this case yields

$$
P_d = \frac{1}{2}(4\pi)^{2+(d-(d+2)N)/2}E^{dN-d-2}\frac{\Gamma(d/2)^N}{\Gamma(Nd/2)\Gamma((N-1)d/2)} \tag{6}
$$

This is easily confirmed for $N=2$ to be correct for all (even odd) d.