Conventions and Useful Formulae

I. *Metric* (“Spacelike” Convention)

$$\eta_{kl} = \delta_{kl} \qquad \eta_{00} = -1 \qquad \eta_{0k} = 0,$$

so that

$$A \cdot B = \eta_{\mu\nu} A^\mu B^\nu = \mathbf{A} \cdot \mathbf{B} - A^0 B^0.$$

This convention is the same as Srednicki, but the **opposite** of Peskin and Schroeder (PS): their metric is $g_{\mu\nu} = -\eta_{\mu\nu}$. Thus $(A \cdot B)_{PS} = -A \cdot B$. We always raise and lower indices with $\eta_{\mu\nu}$ or $\eta^{\mu\nu}$. The “natural” forms x^μ, ∂_μ will be identical to PS, but the “unnatural” forms x_μ, ∂^μ will be the negative of the PS ones. For example, the quantum mechanical association of energy and momentum with time and space derivatives, for us, reads

$$p^\mu = \frac{1}{i}\partial^\mu = -i\partial^\mu$$

and the standard plane wave reads $e^{ik\cdot x}$. Generally, our conventions look most like those of quantum mechanics!

Units:

$$\hbar = c = 1 \qquad \alpha = \frac{e^2}{4\pi}.$$

We, however shall take $e > 0$, the positron charge, so that the electron charge is $-e$.

II. *Dirac Matrices* γ^μ, $\mu = 0, 1, 2, 3$.

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$$

where the braces denote the anticommutator. Sometimes we use $\beta \equiv \gamma^0$ and $\alpha^i = \gamma^0\gamma^i$. These conventions are identical to PS.

It is convenient to define

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

and

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

where the square brackets denote the commutator. The sixteen matrices 1, γ_5, γ^μ, $\gamma_5\gamma^\mu$, $\sigma^{\mu\nu}$ are complete in the sense that every 4×4 matrix can be written as a linear combination of them. These definitions are in agreement with those of PS.

Representations of γ^μ

Let σ^k, $k = 1, 2, 3$ be the 2×2 Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Standard Representation:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \sigma^{km} = \epsilon_{kmn} \begin{pmatrix} \sigma^n & 0 \\ 0 & \sigma^n \end{pmatrix} \qquad \sigma^{k0} = -i \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}$$

where ϵ_{kmn} is the completely antisymmetric three tensor with $\epsilon_{123} = +1$.

Chiral Representation

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \qquad \sigma^{km} = \epsilon_{kmn} \begin{pmatrix} \sigma^n & 0 \\ 0 & \sigma^n \end{pmatrix} \qquad \sigma^{k0} = i \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}$$

This is also sometimes called the natural representation, because the Lorentz generators $\sigma^{\mu\nu}$ are block diagonal with 2×2 blocks.

III. **Dirac Equation**

$$\frac{1}{i}\gamma^\mu \partial_\mu \psi + m\psi = 0$$

[This equation *implies* $(-\partial^2 + m^2)\psi = 0$.]

Plane Wave Solutions

$$\begin{aligned} \text{Positive frequency}: \quad & \psi_p^+(x) = u(p)e^{ip\cdot x} \\ \text{Negative frequency}: \quad & \psi_p^-(x) = v(p)e^{-ip\cdot x} \end{aligned}$$

where $p^0 \equiv \omega_m(\mathbf{p}) \equiv E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2} > 0$ in both cases. u and v satisfy the algebraic equations:

$$(\gamma \cdot p + m)u(p) = 0$$
$$(\gamma \cdot p - m)v(p) = 0$$

and we can relate v to u in a standard way:

$$v(p) \equiv i\gamma^2 u^*(p)$$

where $*$ denotes complex conjugation. A convenient normalization for u is

$$u^\dagger(p)u(p) = 2\sqrt{\mathbf{p}^2 + m^2}$$

Dirac Adjoint:

$$\bar{\psi} \equiv \psi^\dagger \gamma^0$$

where $\dagger$ denotes the usual adjoint: transpose plus complex conjugation. The γ matrices are invariant under the Dirac adjoint:

$$\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu.$$

Thus $\bar{\psi}$ and $\bar{u}$ satisfy

$$\bar{\psi}(x)(i\overleftarrow{\partial} \cdot \gamma + m) = 0$$
$$\bar{u}(p)(\gamma \cdot p + m) = 0$$

Projection Matrices

Let $u\bar{u}$, $v\bar{v}$ denote the 4×4 matrices with elements $u^\alpha \bar{u}^\beta$, $v^\alpha \bar{v}^\beta$ respectively, then

$$u\bar{u} = \frac{1}{2}(m - \gamma \cdot p)(1 - \gamma_5 \gamma \cdot s)$$
$$v\bar{v} = \frac{1}{2}(-m - \gamma \cdot p)(1 - \gamma_5 \gamma \cdot s)$$

where either

$$s^0 = \frac{\mathbf{p} \cdot \hat{\mathbf{s}}}{m}$$
$$\mathbf{s} = \hat{\mathbf{s}} + \mathbf{p}\frac{\mathbf{p} \cdot \hat{\mathbf{s}}}{m(m + \omega)}$$

and $\hat{\mathbf{s}}$ is a unit vector in the direction of the polarization in the rest frame, *i.e.*

$$\hat{\mathbf{s}} \cdot \sigma u_{\hat{\mathbf{s}}}(0) = u_{\hat{\mathbf{s}}}(0),$$

or, if spins are labelled by helicity,

$$s^0 = 2h\frac{|\mathbf{p}|}{m}$$
$$\mathbf{s} = 2h\frac{\omega\mathbf{p}}{m|\mathbf{p}|}.$$

Note: in the case of zero mass the projectors have a smooth limit only for the helicity basis:

$$u_h\bar{u}_h = -\frac{1}{2}(1+2h\gamma_5)\gamma\cdot p$$
$$v_h\bar{v}_h = -\frac{1}{2}(1-2h\gamma_5)\gamma\cdot p.$$

IV. Differential Cross Section and Decay Rate:

$$d\sigma = \frac{d^3p'_1\ldots d^3p'_N}{\prod_i[(2\pi)^3 2E'_i]}(2\pi)^4\delta^{(4)}(\sum_i p'_i - p_1 - p_2)\frac{1}{4E_1E_2|\mathbf{v_1}-\mathbf{v_2}|}|\mathcal{M}|^2$$
$$d\Gamma = \frac{d^3p'_1\ldots d^3p'_N}{\prod_i[(2\pi)^3 2E'_i]}(2\pi)^4\delta^{(4)}(\sum_i p'_i - p)\frac{1}{2m}|\mathcal{M}|^2$$

N. B. If there are groups of identical particles in the final state, a statistical factor $1/r!$ must be included for each group of r identical particles when calculating total cross sections and total rates.

V. Useful Identities Involving Dirac Matrices

$$\mathrm{Tr}[\text{odd number of } \gamma'\text{s}] = 0$$
$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = -4\eta^{\mu\nu}$$
$$\mathrm{Tr}[\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu] = 4[\eta^{\kappa\lambda}\eta^{\mu\nu} - \eta^{\kappa\mu}\eta^{\lambda\nu} + \eta^{\kappa\nu}\eta^{\lambda\mu}]$$
$$\mathrm{Tr}[\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_5] = -4i\epsilon^{\kappa\lambda\mu\nu}$$

where $\epsilon^{\kappa\lambda\mu\nu}$ is completely antisymmetric and $\epsilon^{0123} = +1$.

$$\gamma^\mu\gamma^\nu\gamma^\rho = -i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5 - \eta^{\mu\nu}\gamma^\rho + \eta^{\mu\rho}\gamma^\nu - \eta^{\nu\rho}\gamma^\mu$$
$$\gamma_\mu\gamma^\lambda\gamma^\mu = 2\gamma^\lambda$$
$$\gamma_\mu\gamma^\kappa\gamma^\lambda\gamma^\mu = 4\eta^{\kappa\lambda}$$
$$\gamma_\mu\gamma^\kappa\gamma^\lambda\gamma^\rho\gamma^\mu = 2\gamma^\rho\gamma^\lambda\gamma^\kappa$$
$$\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = -2i\gamma_5\sigma^{\mu\nu}$$

Note the occasional sign differences of these identities compared to PS, (A.27), (A.29), due to our opposite sign metric. Note that PS, (A.30) are valid as written with our conventions.