Quantum Field Theory

Problem Set 1

Due: 18 January 2019

Reading: S, Sections 1-3; Lecture Notes, Chapter 1.

This and all future homework will be posted on the course webpage:

http://www.phys.ufl.edu/∼thorn/homepage/qftinfo.html

In the homework assignments, I will refer to problems in Srednicki's book by prefixing the problem number with S. Note that there are many problems in his book which ask the reader to fill in gaps in the preceding discussion (e.g. S 2.1 through 2.7). I will generally not assign such problems to be graded, but you should nonetheless make sure you understand how to do them.

1. **Lorentz Invariance**

 a) Show by direct calculation that the Lorentz boost in the 1 direction leaves the Minkowski scalar product invariant.

 b) For a general Lorentz transformation, specified by the matrix ${\Lambda^\mu}_\nu$, prove that $(\Lambda^0_0)^2 \geq 1$.

2. **Lorentz commutator algebra** Consider the $\boldsymbol{x}$ and $\boldsymbol{p}$ operators for a quantum particle. The generators of rotations are the components of angular momentum $\boldsymbol{J} = \boldsymbol{x} \times \boldsymbol{p}$. In class we constructed the Lorentz boost generators

$$\boldsymbol{K} = -(\boldsymbol{x}\sqrt{\boldsymbol{p}^2 + m^2} + \sqrt{\boldsymbol{p}^2 + m^2}\boldsymbol{x})/2 + \boldsymbol{p}t.$$

The Lorentz algebra is:

$$[J^k, J^l] = i\hbar\epsilon_{klm}J^m, \qquad [K^k, J^l] = i\hbar\epsilon_{klm}K^m, \qquad [K^k, K^l] = -i\hbar\epsilon_{klm}J^m.$$

The first commutator is known from basic quantum mechanics, and the second one is a consequence of the fact that $\boldsymbol{K}$ is a vector operator. Using the canonical commutations relations $[x^k, p^l] = i\hbar\delta_{kl}$, prove the third commutator, which is new. Also transcribe these commutators to covariant notation using $M_{ij} = \epsilon_{ijk}J^k$ and $M_{0i} = K^i$ to form the tensor $M_{\mu\nu}$ confirming Eq. (1.17) in the Lecture Notes.

3. S, Problem 2.9, part d) only.

4. **The Scalar Field**

 a)Show that the Klein-Gordon scalar wave equation is invariant under a Lorentz trasformation if the field ϕ transforms as a scalar field, *i.e.* $\phi'(x') = \phi(x)$. Here $x'^\mu = \Lambda^\mu_\nu x^\nu$ is a Lorentz transformation.

b) Show by direct substitution of the continuum field expansions

$$\phi(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\hbar}{2\omega(\boldsymbol{k})}}(a(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + a^\dagger(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}) \tag{1}$$

$$\pi(\boldsymbol{x}) = -i\int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\hbar\omega(\boldsymbol{k})}{2}}(a(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} - a^\dagger(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}) \tag{2}$$

into the formulas for the energy and momentum of the continuum scalar field that

$$H_\phi - E_0 = \int d^3k\hbar\omega(\boldsymbol{k})a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) \tag{3}$$

$$\boldsymbol{P} = -\int d^3x\pi(\boldsymbol{x})\nabla\phi(\boldsymbol{x}) = \int d^3k\hbar\boldsymbol{k}a^\dagger(\boldsymbol{k})a(\boldsymbol{k}). \tag{4}$$

Note that in the future we will be assuming units where $\hbar = 1$. I left $\hbar \neq 1$ in this problem to show the familiar Planck condition $E = \hbar\omega = h\nu$ and the de Broglie relation $\boldsymbol{P} = \hbar\boldsymbol{k}$. A difference between Srednicki's and my conventions is that I normalize creation and annihilation operators so that $[a, a^\dagger] = \delta$, compared to his $[a, a^\dagger] = 2\omega(\boldsymbol{k})(2\pi)^3\delta$. (see S (3.29)) This explains the apparent difference between the above equations and S (3.19).