Quantum Field Theory

Problem Set 2

Due: Friday, 25 January 2019

Reading: Lecture Notes, Chapters 1,2; S, Sections 54-55.

5. Starting with the expansion of $\phi(x), \pi(x)$ in terms of creation and annihilation operators, show that the canonical commutation rules

$$\begin{aligned} [\phi(\boldsymbol{x},t),\phi(\boldsymbol{y},t)] &= [\pi(\boldsymbol{x},t),\pi(\boldsymbol{y},t)] = 0 \\ [\phi(\boldsymbol{x},t),\pi(\boldsymbol{y},t)] &= i\hbar\delta(\boldsymbol{x}-\boldsymbol{y}) \end{aligned} \tag{1}$$

follow from $[a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})] = \delta(\boldsymbol{p}-\boldsymbol{q})$ and $[a(\boldsymbol{p}), a(\boldsymbol{q})] = 0$.

6. Recall that a Lorentz transformation of a scalar field is $\phi'(x) = \phi(\Lambda^{-1}x)$. In QFT, Lorentz transformations are represented by unitary operators $U(\Lambda)$ which accordingly transform the scalar field as

$$\phi'(x) \equiv U^{-1}(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x) \tag{2}$$

Show that this requirement implies that $a, a^\dagger$ transform as

$$U(\Lambda)^{-1}a(\boldsymbol{k})U(\Lambda) = \sqrt{\frac{\omega(\Lambda^{-1}\boldsymbol{k})}{\omega(\boldsymbol{k})}}a(\Lambda^{-1}\boldsymbol{k}), \qquad U(\Lambda)^{-1}a^\dagger(\boldsymbol{k})U(\Lambda) = \sqrt{\frac{\omega(\Lambda^{-1}\boldsymbol{k})}{\omega(\boldsymbol{k})}}a^\dagger(\Lambda^{-1}\boldsymbol{k})$$

[The notation $\Lambda^{-1}\mathbf{k}$ means the spatial components of the four vector $(\Lambda^{-1})^\mu{}_\nu k^\nu$, where $k^0 \equiv \sqrt{\mathbf{k}^2+m^2}$. Note that Srednicki chooses a different normalization for $a, a^\dagger$: $[a_S(\boldsymbol{q}), a_S^\dagger(\boldsymbol{p})] = (2\pi)^3 2\omega(\boldsymbol{p})\delta(\boldsymbol{q}-\boldsymbol{p})$.] Show that with his normalization the square root prefactor in the transformation law is absent.

7. **The Free Complex Scalar Field**. A non-hermitian ("complex") scalar field has the expansion

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(\boldsymbol{p})}}\left(a(\boldsymbol{p})e^{ip\cdot x} + b^\dagger(\boldsymbol{p})e^{-ip\cdot x}\right), \qquad \phi^\dagger \neq \phi \tag{3}$$

Here $p\cdot x \equiv \boldsymbol{p}\cdot\boldsymbol{x} - \omega(\boldsymbol{p})t$; $a^\dagger(\boldsymbol{p})$ creates a spin 0 particle; and $b^\dagger(\boldsymbol{p})$ creates the associated antiparticle. Their commutation relations are

$$[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = [b(\boldsymbol{p}), b^\dagger(\boldsymbol{p}')] = \delta(\boldsymbol{p}'-\boldsymbol{p}),$$

with all other commutators vanishing.

(a) Show from (3) that ϕ and $\dot{\phi}^\dagger$ satisfy the equal time commutation relations

$$[\phi(\boldsymbol{x},t), \dot{\phi}^\dagger(\mathbf{y},t)] = i\delta^3(\boldsymbol{x}-\mathbf{y}),$$

and that ϕ satisfies the Klein-Gordon Equation

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2} - m^2\right)\phi(x) = 0.$$

(b) Alternatively we can work with hermitian fields ϕ_1, ϕ_2 defined by $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. Show that

$$[\phi_k(\boldsymbol{x},t), \dot{\phi}_l(\mathbf{y},t)] = i\delta_{kl}\delta^3(\boldsymbol{x}-\mathbf{y}).$$

Thus $\dot{\phi}_k \equiv \pi_k$ is the conjugate momentum to ϕ_k, and the Hamiltonian is just the sum of two commuting terms, one for each of the hermitian fields ϕ_1, ϕ_2:

$$H = \int d^3x \frac{1}{2}\sum_k (\pi_k^2 + (\nabla\phi_k)^2 + m^2\phi_k^2).$$

(c) Now returning to the original non-hermitian field ϕ, and defining $\pi = (\dot{\phi}_1^\dagger - i\dot{\phi}_2^\dagger)/\sqrt{2}$, show that

$$H = \int d^3x(\pi\pi^\dagger + \nabla\phi^\dagger\nabla\phi + m^2\phi^\dagger\phi),$$

and the equal time commutation relations become

$$[\phi(\boldsymbol{x},t), \pi(\boldsymbol{y},t)] = i\delta^3(\boldsymbol{x}-\mathbf{y}). \quad (4)$$

8. **Coupling Classical Electromagnetism to a Quantum Scalar Field**

(a) The minimal substitution rule for coupling an external electromagnetic field in a gauge invariant way is the substitution $\partial \to \partial - iqA$ in the Klein-Gordon equation. [Note that the appearance of i in this rule is why electromagnetism must couple to a complex (*i.e.* non-hermitian field).] Show that the resulting equation follows from the Heisenberg equations derived, using (4) and the Hamiltonian

$$H = \int d^3x(\pi\pi^\dagger + (\nabla + iq\mathbf{A})\phi^\dagger(\nabla - iq\mathbf{A})\phi + m^2\phi^\dagger\phi + iqA_0(\pi\phi - \phi^\dagger\pi^\dagger)),$$

(b) We shall soon learn that gauge invariant time evolution implies that a conserved current can be defined in terms of the change in the Schrödinger picture Hamiltonian, under a small change in the potentials with canonical variables fixed (i.e. ϕ, its *spatial* derivatives and π are held fixed):

$$U^\dagger(t)\delta H_S U(t) = -\int d^3x j_\mu(\boldsymbol{x},t)\delta A^\mu(\boldsymbol{x},t).$$

Here $U(t)$ converts Schrödinger to Heisenberg picture. From the Hamiltonian in part (a) use this principle and the Heisenberg equations to obtain the expression for the current

$$j_\mu(x) = -iq(\phi^\dagger \partial_\mu \phi - (\partial_\mu \phi^\dagger)\phi) - 2q^2 A_\mu \phi^\dagger \phi.$$

Confirm that $\partial_\mu j^\mu = 0$ as a consequence of the Klein-Gordon equation coupled to A_μ.

(c) Work out the charge $Q = \int d^3x j^0$ in terms of creation and annihilation operators for the case of zero external field ($A_\mu = 0$).