Quantum Field Theory

Problem Set 3

Due: 1 February 2019

Reading: Lecture Notes, Chapter 1; S, Sections 54-55.

9. Show that in Coulomb gauge, the canonical free EM Hamiltonian reduces to

$$H_{Coulomb} = \int d^3x \frac{1}{2} \sum_{l=1}^{3} \left[\Pi^l \Pi^l + \nabla A^l \cdot \nabla A^l \right] \tag{1}$$

where the l sum is just over the components of the vector operators $\mathbf{\Pi}, \boldsymbol{A}$. Then show, by substituting the expansions for Π^l, A^l in terms of creation and annihilation operators into this formula, that the Hamiltonian is just $H_{Coul} = \int d^3k |\boldsymbol{k}| \sum_l a_l^\dagger(\boldsymbol{k}) a_l(\boldsymbol{k})$ and also that the total momentum is $\boldsymbol{P} = \int d^3k \boldsymbol{k} \sum_l a_l^\dagger(\boldsymbol{k}) a_l(\boldsymbol{k})$.

10. **Canonical Energy Momentum Tensor**. Given a general Lagrangian density $\mathcal{L}(\partial_\mu \phi_k, \phi_k)$, the canonical energy momentum tensor is defined by

$$T^{\mu\nu}_{\rm Can} \;=\; -\sum_k \partial^\mu \phi_k \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_k)} + \eta^{\mu\nu} \mathcal{L} \tag{2}$$

a) Use the field equations implied by Hamilton's principle, applied to $S = \int d^4x \mathcal{L}$, to prove that this construction is automatically conserved: $\partial_\nu T^{\mu\nu} = 0$.

b) While the canonical construction is automtically conserved, it is *not* automatically symmetric. Construct the canonical energy momentum tensor for the free electromagnetic field and show that it is not symmetric, i.e. $T^{\mu\nu}_{\rm Can} \neq T^{\nu\mu}_{\rm Can}$

c) In class we used a definition of the EM energy momentum tensor which is symmetric. Show that it differs from the canonical one, defined above, by a term that does not contribute to the total energy and momentum (when the field equations are satisfied). In other words, show that $T^{\mu 0}_{\rm Can} - T^{\mu 0}_{\rm Class}$ is a total *spatial* derivative. The symmetric energy momentum tensor is sometimes called the "improved" energy momentum tensor.

11. In class we obtained the angular momentum of the EM field

$$\boldsymbol{J} \;=\; \int d^3x \sum_k E_k (\boldsymbol{x} \times \boldsymbol{\nabla}) A_k + \int d^3k a_b^\dagger(\boldsymbol{k}) a_a(\boldsymbol{k}) \boldsymbol{S}_{ab} \tag{3}$$

where the 2×2 photon 'spin" matrix is given by $\boldsymbol{S}_{ab} = i \boldsymbol{\epsilon}_a \times \boldsymbol{\epsilon}_b^*$.

a) Evaluate the action of the first "orbital" term on a single photon state $a_c^\dagger(\boldsymbol{k})|0\rangle$ and show that it doesn't contribute to the photon helicitty.

b) For $\boldsymbol{k}$ parallel to the z-axis $\boldsymbol{k} = k\hat{z}$ and the choices $\epsilon_1 = (1, i, 0)/\sqrt{2}$ and $\epsilon_2 = (1, -i, 0)/\sqrt{2}$, calculate the three matrices S^x, S^y, S^z.