Quantum Field Theory

Problem Set 4

Due: 8 February 2019

Reading: S, 33-40; Lecture Notes, Chapter 2,3.

12. **Representations for γ^μ.** For this exercise we consider representations of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = -2\eta_{\mu\nu}$ for general spcaetime dimension D: so there are D anticommuting matrices γ^μ, $\mu = 0, 1, \ldots, D-1$. Remember that γ^0 must be hermitian and the spatial components γ^k must be antihermitian.

a) For D even, show that the matrix $\gamma_{D+1} \equiv i\gamma^0\gamma^1 \cdots \gamma^{D-1}$ anticommutes with all of the γ^μ. This means that given a representation of γ^μ for D even, one can treat γ_{D+1} or $i\gamma_{D+!}$ as an extra spatial γ. Show that it is hermitian for $D = 4$ but antihermitian for $D = 2, 6$.

b) Show that in D space-time dimensions the γ matrices must be at least of size $2^{D/2} \times 2^{D/2}$ for D even and at least $2^{(D-1)/2} \times 2^{(D-1)/2}$ for D odd.

c) In addition to the standard and chiral representations, it is sometimes useful to have a Majorana representation, for which all gamma matrices are imaginary $\gamma^{\mu *} = -\gamma^\mu$. Find such a representation for $D = 4$, by selecting 4 of the 5 anticommuting matrices γ^μ, γ_5 in the standard representation and, if necessary, mutiplying some of them by i.

13. **Lorentz Covariance of Dirac Equation** Let $\Lambda^\mu{}_\nu$ represent a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. In the new frame, the Dirac equation is

$$\frac{1}{i}\gamma^\mu \partial'_\mu \psi'(x') + m\psi'(x') = 0$$

Note that the *same* gamma matrices are used in every Lorentz frame! Our goal is to express $\psi'(x')$ in terms of $\psi(x)$.

(a) Let $\sigma^{mn} = \frac{1}{2}i[\gamma^m, \gamma^n]$, where $m, n = 1, 2, 3$ are spatial indices. Using the standard or chiral representation, show that $\sigma^{mn} = \epsilon_{mnk}\Sigma^k$ where $\mathbf{\Sigma}/2$ is the spin matrix for the Dirac equation.

(b) This result suggests that $M^{\mu\nu} = \frac{1}{4}i[\gamma^\mu, \gamma^\nu] \equiv \frac{1}{2}\sigma^{\mu\nu}$ is the generator of Lorentz transformations. Show that $M^{\mu\nu}$ satisfies the Lorentz algebra commutation relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\rho\nu} - \eta^{\nu\sigma}M^{\rho\mu}), \quad (1)$$

It will be helpful to first show that the commutators of M with γ are those with γ a four-vector, *i.e.*

$$[M^{\mu\nu}, \gamma^\rho] = i(\eta^{\mu\rho}\gamma^\nu - \eta^{\nu\rho}\gamma^\mu).$$

(c) Exploit these results to show that

$$\psi'(x') = e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\psi(\Lambda^{-1}x')$$

is a solution of the Dirac equation in the new frame provided $\psi(x)$ is a solution in the original frame and the matrix λ is related to Λ by

$$\Lambda^{\mu}{}_{\nu} = (e^{-\lambda})^{\mu}{}_{\nu}.$$

[Note that because of the group property of the $\sigma^{\mu\nu}$ established in the second part of this problem, it is sufficient to demonstrate this for infinitesimal λ.]

14. **Helicity basis for spin 1/2 particles.** Helicity is defined as the component of angular momentum of a particle along its momentum, *i.e.* $h = \boldsymbol{p}\cdot\boldsymbol{J}/|\boldsymbol{p}|$ with $\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{p} + \boldsymbol{\sigma}/2$. Note that since $\boldsymbol{p}\cdot(\boldsymbol{r}\times\boldsymbol{p}) = 0$, we may write, more simply, $h = \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}/2$, where $\hat{\boldsymbol{p}}$ is a unit vector parallel to $\boldsymbol{p}$.

(a) Show that the eigenvalues of h are $\pm 1/2$.

(b) Consider a single particle state $|\boldsymbol{p}, h\rangle$ with momentum $\boldsymbol{p}$ and helicity h. Show that rotations leave the helicity of the state unchanged.

(c) For $\boldsymbol{p} = p\hat{\mathbf{z}}$, $h = \sigma_z/2$ and we may take $|p\hat{\mathbf{z}}, h = \pm 1/2\rangle \equiv |p\hat{\mathbf{z}}, \sigma_z = \pm 1\rangle$ and we fix the ambiguity at $\boldsymbol{p} = 0$ by defining $|\mathbf{0}, h = \pm 1/2\rangle \equiv |\mathbf{0}, \sigma_z = \pm 1\rangle$. According to (b), we may define

$$|\boldsymbol{p}, h = \pm 1/2\rangle \equiv R_0(\boldsymbol{p})|p\hat{\mathbf{z}}, \sigma_z = \pm 1\rangle$$

where $R_0(\boldsymbol{p})$ is a standardized rotation that takes $p\hat{\mathbf{z}}$ into $\boldsymbol{p}$. Let θ, ϕ be the polar angles of $\boldsymbol{p}$. Then take

$$R_0(\boldsymbol{p}) \equiv e^{-i\phi J_z} e^{-i\theta J_y} e^{+i\phi J_z}.$$

With this definition and using spinor notation

$$|\boldsymbol{p}, \sigma_z = 1\rangle \equiv |\boldsymbol{p}\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\boldsymbol{p}, \sigma_z = -1\rangle \equiv |\boldsymbol{p}\rangle \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

show that

$$|\boldsymbol{p}, h = 1/2\rangle = |\boldsymbol{p}\rangle \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \equiv |\boldsymbol{p}\rangle \chi_{1/2}(\boldsymbol{p}) \tag{2}$$

$$|\boldsymbol{p}, h = -1/2\rangle = |\boldsymbol{p}\rangle \begin{pmatrix} -e^{-i\phi}\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} \equiv |\boldsymbol{p}\rangle \chi_{-1/2}(\boldsymbol{p}). \tag{3}$$

(d) Using the results of (c) prove the following identities:

$$\chi_\lambda(\boldsymbol{p}) = -ie^{i\lambda\pi + 2i\lambda\phi}\chi_{-\lambda}(-\boldsymbol{p}) \tag{4}$$

$$i\sigma^2\chi_\lambda(\boldsymbol{p}) = e^{2i\lambda\phi}\chi^*_\lambda(-\boldsymbol{p}) \tag{5}$$

15. Our discussion of the anticommutators of the creation and annihilation operators used in second quantizationi in class was a bit heuristic. Show that the concrete definition of $b_\alpha^\dagger$ acting on the occupatioin number basis,

$$b_\alpha^\dagger|n_1, n_2, \ldots, n_\alpha, \ldots\rangle \;\equiv\; (-)^{\sum_{\beta<\alpha} n_\beta}|n_1, n_2, \ldots, 1+n_\alpha, \ldots\rangle \tag{6}$$

where it is understood that $|n_1, n_2, \ldots\rangle \equiv 0$ if any $n_\beta > 1$. implies that $\{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha\beta}$ and all other anticommutators are zero.