Quantum Field Theory

Problem Set 5

Due: Friday, 15 February 2019

Reading: Lecture Notes, Chapters 2-3. S, Sections 33-36,40.

16. In problem 13 of Problem Set 4, you showed that the Lorentz transformation of the Dirac field is $\psi(x) \to e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\psi(\Lambda^{-1}x)$ and in class we learned that the parity transformation is $\psi(\boldsymbol{x},t) \to e^{i\phi}\gamma^0\psi(-\boldsymbol{x},t)$. Show from these rules that the Dirac bilinears, $\bar{\psi}(x)\psi(x)$, $\bar{\psi}(x)\gamma_5\psi(x)$, $\bar{\psi}(x)\gamma^\mu\psi(x)$, $\bar{\psi}(x)\gamma_5\gamma^\mu\psi(x)$, and $\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$, transform under Lorentz and parity transformations as scalar, pseudoscalar, vector, axial vector, and second rank tensor fields respectively. Here $\bar{\psi} \equiv \psi^\dagger\gamma^0$ is the Dirac adjoint. A useful first step in the solution is to prove that $\bar{\psi}(x) \to \bar{\psi}(\Lambda^{-1}x)e^{+i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}$ under Lorentz transformations. Also it is enough to specialize to infinitesimal Lorentz transformatioins. (The terms pseudo- and axial refer to opposite than "normal" parity properties. For a normal 4-vector, the time component is even and the space component is odd under parity. A normal scalar is even under parity.)

17. Show that the Dirac equation, regarded as a single particle Schrödinger equation, implies that the free electron has a Landé g factor of 2 ($\boldsymbol{\mu} \equiv g(Q/(2mc)\boldsymbol{S})$. [Hint: Identify the magnetic moment by examining the Dirac equation for a weak slowly varying magnetic field $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ in the limit that the electron is moving slowly. We assume that the magnetic field field enters the Dirac equation via the minimal substitution $\nabla \to \nabla - iq\boldsymbol{A}$. In this situation, after eliminating the lower two components of $\psi = \binom{\phi}{\chi}$ in favor of the upper two components in standard representation, the D.E. should go over to the NR Schrodinger equation with a term that can be identified as $-\boldsymbol{\mu}\cdot\boldsymbol{B}$.]

18. Take the Dirac equation for an electron in the Coulomb field of a proton to be

$$\left(i\gamma\cdot\partial + \gamma^0\frac{e^2}{4\pi r} - m\right)\psi = 0.$$

(a) Writing the wave function for a stationary state in the form

$$\psi(x) = e^{-imt-iEt}\begin{pmatrix}\phi(\mathbf{x})\\ \chi(\mathbf{x})\end{pmatrix}$$

where ϕ and χ are 2 component spinors, solve for χ in terms of ϕ and show that ϕ satisfies the equation

$$\left\{-\boldsymbol{\sigma}\cdot\nabla\left(2m + E + \frac{e^2}{4\pi r}\right)^{-1}\boldsymbol{\sigma}\cdot\nabla - \frac{e^2}{4\pi r}\right\}\phi = E\phi.$$

(b) By taking suitable limits on the Dirac equation in this form, find the first relativistic corrections to the nonrelativistic Schrödinger equation. Identify the correction terms with the familiar spin-orbit and relativistic kinetic energy correction terms used to understand the fine structure splittings in non-relativistic quantum mechanics, paying attention to any differences. What can you conclude about the splittings of the 2s and 2p levels of hydrogen?

[Hint: Expand the inverse in the first term on the l.h.s. Remember that in units with $\hbar = c = 1$, $e^2/4\pi \approx 1/137 << 1$, and the Bohr radius of hydrogen $\gg$ the Compton wavelength of the electron, so we may assume E and $e^2/4\pi r$ are both $<< m$.]

19. **Quantum Numbers of Positronium.**

a)A state of positronium (*i.e.* a hydrogenic e^+e^- atom) can be written as

$$|\Psi\rangle = \int d^3p \sum_{\mu_1\mu_2} F(\vec{p}, \mu_1; -\vec{p}, \mu_2) b^\dagger_{\vec{p}\mu_1} d^\dagger_{-\vec{p}\mu_2} |0\rangle.$$

where it is convenient to label spin not by helicity but by the spinor basis $\phi_\mu = \binom{1}{0}, \binom{0}{1}$ which are just eigenstates of σ^z, so that parity does not touch spin. When F corresponds to orbital angular momentum L and total spin $S(S = 0$ or $1)$ how does $|\Psi\rangle$ transform under parity and charge conjugation?

(b) It is a fact that an n photon state has charge conjugation eigenvalue $C = (-)^n$. If C is conserved in the spontaneous annihilation of positronium into photons, what is the minimum number of photons in the final state of the decay of the ground states of ortho-(S=1) and para-(S=0) positronium?