Quantum Field Theory

Problem Set 6

Due:Friday, 22 February 2019

Reading:Lecture Notes, Chapter 3,4,5.

20. Revisit our definition of the charge conjugation transformation of the Dirac field,, but this time using a Majorana representation where all the γ^μ are imaginary. Show that in this representation there is no need for a matrix like $i\gamma^2$ that we needed in the standard representation. Also confirm that the Chiral representation does require this matrix.

21. As mentioned in class, any system of spin 1/2 fields can be described in terms of a collection of Weyl fields with the same chirality, say $L_k(x)$. Then the most general mass term can be expressed in the form

$$\sum_{kl} \frac{m_{kl}}{2} L_k^T i\gamma^2 \beta L_l + \sum_{kl} \left(\frac{m_{kl}}{2} L_k^T i\gamma^2 \beta L_l \right)^\dagger . \tag{1}$$

where the complex matrix m is symmetric $m^T = m$. A unitary transformation $L_k \to U_{km} L_m$ leaves the kinetic term $\int d^3x \sum_k L_k^\dagger (-i\boldsymbol{\alpha} \cdot \nabla) L_k$ invariant.

a) Show that the change of variables $L = UL'$, with $U^\dagger U = I$, modifies the mass matrix to $m' = U^T m U$. Note that this is *not* a similarity transformation when U is complex!

b) Show that one can choose U so that m' is a diagonal matrix $m'_{kl} = m_k \delta_{kl}$. *Hint*: first note that there is a unitary V which diagonalizes the hermitian matrix $m^\dagger m$. Then prove that the real and imaginary parts of $V^T m V$ commute with each other, and so can be simultaneously diagonalized by a real orthogonal matrix similarity transformation..

c) In the case of 2 fields L_k, $k = 1, 2$, apply this procedure to diagonalize the mass matrix $\begin{pmatrix} 0 & m \\ m & M \end{pmatrix}$, where m, M are both allowed to be complex. Discuss what happens to the mass eigenvalues when $|M| \gg |m|$. *HInt*: Start by multiplying the mass matrix on the left and right by a diagonal matrix with entries $e^{i\alpha}, e^{i\beta}$ and choose α, β so the resulting matrix is real, and hence hermitian.

22. **Time reversal and charge conjugation**

a) Derive how the Dirac bilinears, $\bar{\psi}(x)\psi(x)$, $\bar{\psi}(x)i\gamma_5\psi(x)$, $\bar{\psi}(x)\gamma^\mu\psi(x)$, $\bar{\psi}(x)\gamma_5\gamma^\mu\psi(x)$, and $\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$ transform under time reversal when ψ is the second quantized Dirac field operator.

(b) How do these same bilinears transform under charge conjugation.

23. Combine the transformations of parity P charge conjugation C and time reversal on the bilinears, which you derived in problem 16 (from set 5) and in problem 22, in all possible ways, CP, CT, PT, and CPT, and detemine how the various bilinears transform under each combination. Thus you will confirm the results quoted in section 3.6 of our Lecture Notes.