Quantum Field Theory

Problem Set 7

Due: 1 March 2019

Reading: Lecture Notes, Chapters 4-5

24. The results of problem 16 (Problem Set 5) are based on the fact that the similarity transformation

$$e^{i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\Gamma e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4} \tag{1}$$

with Γ any of the matrices $I, i\gamma_5, \gamma^\mu, \gamma_5\gamma^\mu, \sigma^{\mu\nu}$ performs a Lorentz transformation on each four-vector index. As noted in part a) below, the matrices

$$e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4} \tag{2}$$

give us the (nonunitary) representation $D(1/2,0)\oplus D(0,1/2)$ of the Lorentz group.

a) Show that the Lorentz generator matrices $M^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$, in the natural (chiral) representation, take the block diagonal forms

$$M^{kl} = \frac{1}{2}\epsilon_{klm}\begin{pmatrix}\sigma^m & 0\\ 0 & \sigma^m\end{pmatrix} \tag{3}$$

$$M^{0k} = \frac{i}{2}\begin{pmatrix}-\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix} \tag{4}$$

This shows that the Lorentz group has two distinct two dimensional representations called $D(1/2,0)$ represented by the upper 2×2 block and $D(0,1/2)$ represented by the lower 2×2 block. The Dirac representation is reducible into the direct sum of these.

b) Show that the matrices

$$e^{i\lambda_{\mu\nu}\sigma^{T\mu\nu}/4}$$

are similar to the matrices (2). (Recall that a similarity transformation on a matrix A is $S^{-1}AS$ for an invertible matrix S. Simply find an S that does the trick.)

25. From the previous problem the transformation (1) may be viewed as belonging to the

$$(D(1/2,0)\oplus D(0,1/2))\otimes(D(1/2,0)\oplus D(0,1/2)) \tag{5}$$

representation of the Lorentz group: Think of the matrix elements Γ_{ab} as a two-index bi-spinor, for which the Lorentz transformation reads

$$\Gamma'_{ab} = \Gamma_{cd}(e^{i\lambda_{\mu\nu}\sigma^{T\mu\nu}/4})_{ca}(e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4})_{db}$$

a) Find the decomposition of the tensor product representation (5) into irreducible representations of the Lorentz group. Note that the decomposition follows the same rules as for $SU(2)$, namely

$$\frac{1}{2} \otimes 0 = \frac{1}{2}, \qquad \frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0.$$

b) By considering the dimensionalities of the representations, relate the results of part (b) to the transformation properties of the matrices $I, i\gamma_5, \gamma^\mu, \gamma_5\gamma^\mu, \sigma^{\mu\nu}$.

26. In class and also as you have found in the previous problem, the irreducible representation $D(1/2, 1/2)$ of the Lorentz group is identified with the 4-vector representation $p^\mu \to \Lambda^\mu{}_\nu p^\nu$. A concrete way to understand this result is in terms of the 2×2 matrix $P \equiv p^0 I + \boldsymbol{p} \cdot \boldsymbol{\sigma}$. Note that P is Hermitian $P^\dagger = P$ when p^μ is a real 4-vector.

a) Show that $\det P = p^{02} - \boldsymbol{p}^2 = -\eta_{\mu\nu} p^\mu p^\nu$. This shows that a transformation on the matrix P which leaves $\det P$ invariant must induce a four vector Lorentz transformation on p^μ.

b) To keep P hermitian, we should restrict the transformations of P to the form $P \to L^\dagger P L$, where L is a 2×2 matrix. Then this transformation leaves $\det P$ invariant if $|\det L| = 1$. Multiplying L by a phase has no effect on $L^\dagger P L$ so wolog we can take $\det L = 1$, so L is in the group of matrices $SL(2, C)$, the group of complex 2×2 matrices with unit determinant. Show that the matrices representing $D(1/2, 0)$ and $D(0, 1/2)$ have unit determinant, and complete the argument that $D(1/2, 1/2)$ is the 4-vector representation of the Lorentz group.

27. Decompose the representation $D(1/2, 1/2) \otimes D(1/2, 1/2)$ into irreducible representations and, with the result of the previous problem in mind, interpret the individual components in terms of two index 4-tensors of different symmetry types.