Quantum Field Theory

Problem Set 8

Due: 15 March 2019

Reading: Lecture Notes, Chapters 5-6

28. We have taken it for granted that the evolution operator $U(t_2, t_1) = T \exp\left\{-i \int_{t_1}^{t_2} H(t) dt\right\}$ is unitary if $H(t)$ is Hermitian.

a) Prove this by using the differential equation satisfied by $U(t, t_1)$.

b) The result must also follow by expanding U and $U^\dagger$ in power series and collecting together all terms with N $H(t)$'s multiplied together. For each $N > 0$ these terms must cancel to give zero. Check this for $N = 1, 2$. [The ambitious can try to construct a proof for general N but do not waste time on this. It is simple if you hit on the right way, very nasty otherwise.]

29. The frame dependence of the concept of simultaneity raises the issue of whether the concept of time ordering can be compatible with Lorentz invariance: two events can have opposite temporal ordering in different Lorentz frames.

a) Quantum fields are supposed to obey Einstein causality, namely $[A(x), B(y)] = 0$ for $x-y$ spacelike ($(x-y)^2 > 0$). Otherwise spacelike separated experiments could interfere with each other. Argue that, as a consequence of this causality, time ordred products of local field operators are compatible with Lorentz invariance.

b) Show that for field operators $A(x), B(y)$, that

$$\frac{\partial}{\partial t} T[A(\boldsymbol{x}, t)B(\boldsymbol{y}, t')] = T[\dot{A}(\boldsymbol{x}, t)B(\boldsymbol{y}, t')] + \delta(t' - t)[A(\boldsymbol{x}, t), B(\boldsymbol{y}, t)], \qquad \dot{A} \equiv \frac{\partial A}{\partial t} \quad (1)$$

if A or B is bosonic. If both A and B are fermionic the commutator in the last term is replaced by an anticommutator. This relation is important in resolving other apparent violations of Lorentz invariance in time dependent perturbation theory.

30. Using the Hamiltonian of problem 8 of set 2, calculate the scattering ampitude for a single charged scalar particle to first order in a static 4 vector potential $A_\mu(\boldsymbol{x})$. Specialize to the Coulomb potential $\boldsymbol{A} = 0$, $A^0 = e/(4\pi|\boldsymbol{x}|)$, and calculate the differential cross section for this process as a function of the initial particle energy and scattering angle. [Since you are working only to first order in A_μ, the amplitude will be proportional to the matrix element of the current operator with $A_\mu = 0$

$$j_\mu(x) \approx -iq(\phi^\dagger \partial_\mu \phi - (\partial_\mu \phi^\dagger)\phi). \quad (2)$$

Also assume that j^μ is normal ordered so that its vacuum expectation value is zero.]

31. **Electron scattering in a purely magnetic field.**

(a) Show that for any purely magnetic field, the scattering calculated to lowest order does not alter the helicity.

(b) Calculate the differential cross-section for electron scattering in the field of a magnetic dipole when the initial and final momenta are both perpendicular to the dipole axis, to lowest order in perturbation theory (Born approximation). [The vector potential of a dipole $\vec{M}$ is $\nabla \times \left(\vec{M}/r\right)$ so you can find its Fourier transform from that of $1/r$.]