Quantum Field Theory

Problem Set 9

Due: 22 March 2019

Reading: Lecture Notes, Chapters 5-6.

32. Our class discussion of the propagator for the Dirac field left out a lot of details. Following the strategies sketched in the notes give a detailed and complete derivation of

a) Equations (5.95) and (5.96).

b) Equation (5.99) starting from (5.97)-(5.98).

33. In this exercise we study the procedure for obtaining scattering amplitudes from time ordered products for a Dirac field interacting with an external electromagnetic field $A_e^\mu(x)$.

a) Evaluate the amplitude through first order in A_e

$$\langle out|T[\psi_a(x)\bar{\psi}_b(y)|in\rangle \tag{1}$$

in terms of the free Dirac propagagor $S_F = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle$. Notice that the first order correction contains a "disconnected" piece which vanishes because $\langle 0|j^\mu(x)|0\rangle = 0$.

b) Now carry out the Fourier transform

$$\int d^4x d^4y e^{-iq\cdot x+ip\cdot y}\langle out|T[\psi_a(x)\bar{\psi}_b(y)]|in\rangle \tag{2}$$

of your result with respect ro both x and y. You should obtain

$$\frac{-i(m-p\cdot\gamma)}{m^2+p^2}(2\pi)^4\delta(q-p)+\frac{-i(m-q\cdot\gamma)}{m^2+q^2}iQ\gamma\cdot\tilde{A}(q-p)\frac{-i(m-p\cdot\gamma)}{m^2+p^2} \tag{3}$$

where $\tilde{A}(q) = \int d^4xe^{-iq\cdot x}A(x)$. Scattering information is contained in the second term which displays poles in p^2 and q^2 at $-m^2$. That is when p and q are momenta for a particle of mass m. These poles can have positive or negative energy $p^0 = \pm\omega(\boldsymbol{p})$ or $q^0 = \pm\omega(\boldsymbol{q})$. According to F.T. definitions, $q^0 > 0$ ($q^0 < 0$)describes an outgoing (incoming) particle. but $p^0 > 0$ ($p^0 < 0$) corresponds to an incoming (outgoing) particle.

c) Referring to the results proved in problem 32 above, depending on the energy sign choices, show that the residue of the poles involves the amplitude for one of the processes

$$\begin{aligned}
\text{Particle Scattering}: &\quad \bar{u}(\boldsymbol{q})iQ\gamma\cdot Au(\boldsymbol{p})\\
\text{Antiparticle Scattering}: &\quad \bar{v}(-\boldsymbol{q})iQ\gamma\cdot Av(-\boldsymbol{p})\\
\text{Pair Production}: &\quad \bar{u}(\boldsymbol{q})iQ\gamma\cdot Av(-\boldsymbol{p})\\
\text{Pair Annihilation}: &\quad \bar{v}(-\boldsymbol{q})iQ\gamma\cdot Au(\boldsymbol{p})
\end{aligned}$$

where $u, \bar{v}$ describe incoming and $\bar{u}, v$ describe outgoing particles.

34. Prove, without using perturbation theory, that the propagator for the Dirac field ψ in the presence of an external electromagnetic field A_μ,

$$S_A(x,y) \equiv \frac{\langle out|T[\psi_a(x)\bar{\psi}_b(y)]|in\rangle}{\langle out|in\rangle},$$

is a Green's function for the differential operator $[i\gamma\cdot\partial + q\gamma\cdot A - m]$:

$$(-i\gamma\cdot(\partial - iqA) + m)_{ac} S_{A\ cb}(x,y) = -i\delta_{ab}\delta(x-y) \tag{4}$$

You will need the facts that the operator $\psi(x)$ obeys the Dirac equation in the external field $A(x)$, that $d\theta(t)/dt = \delta(t)$ and that the anticommutation relations for $\psi(x), \bar{\psi}(y)$ at $x^0 = y^0$ are not changed by the presence of A. This result gives an alternative starting point for the weak field expansion. Confirm that the first order correction agrees with that you obtained in problem 33a.

35. e^+e^- **Pair production in an external EM field**

a) Write down the lowest order contribution to the amplitude for producing an electron-positron pair in the presence of an external field A_μ. Which Fourier components of $A_\mu(x)$ contribute to pair production to lowest order in qA? What will happen to the vacuum persistence amplitude $\langle out|in\rangle$ if pair production is possible?

b) Calculate the differential probability of producing a pair with momenta $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and unobserved helicities, in terms of $\tilde{A}_\mu$, the Fourier transform of $A_\mu(x)$. [You will need

$$\sum_\lambda u_\lambda(p)\overline{u}_\lambda(p) = m - \gamma\cdot p, \qquad \sum_\lambda v_\lambda(p)\overline{v}_\lambda(p) = -m - \gamma\cdot p$$

and formulas for the trace of products of γ matrices. Remember that $\tilde{A}$ is complex and that $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$.] For the special case $\boldsymbol{q}_1 = \boldsymbol{q}$, $\boldsymbol{q}_2 = -\boldsymbol{q}$, $q_1^0 = q_2^0 = \omega(\boldsymbol{q})$, verify that your answer is positive and that it does not depend on $\tilde{A}^0$, and express it in terms of Fourier components of the electromagnetic field $F_{\mu\nu}$. [This verifies the gauge invariance of the calculation.]