Quantum Field Theory

Problem Set 11

Due: 5 April 2019

Reading: Lecture Notes, Chapters 8 and 10. (Chapter 9 is optional reading on scattering in external fields).

40. S, Problem 11.4

41. We introduced the nonabelian gauge potential A_μ as a matrix valued generalization of the vector potential of QED, and determined its gauge transformation so that $(\partial_\mu - igA_\mu)\psi = D_\mu\psi$ transforms as $D_\mu\psi \to \Omega(x)D_\mu\psi$ under $\psi \to \Omega\psi$, In QED we construct a gauge invariant field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which enters the EM part of the Lagrangian as $-F_{\mu\nu}F^{\mu\nu}/4$.

a) To construct a nonabelian field strength analogous to that in QED, we cannot insist that it be gauge invariant but we can require that $F_{\mu\nu} \to \Omega F_{\mu\nu}\Omega^{-1}$, so that $\mathrm{Tr} F_{\mu\nu}F^{\mu\nu}$ is gauge invariant. Show that if we regard D_μ as a matrix valued differential operator, then under a gauge transformation

$$A_\mu \to \Omega A_\mu \Omega^{-1} - \frac{i}{g}(\partial_\mu\Omega)\Omega^{-1} \tag{1}$$

$D_\mu \to \Omega D_\mu \Omega^{-1}$.

b) From part a) it follows that $D_\mu D_\nu \to \Omega D_\mu D_\nu \Omega^{-1}$ so that it transforms as we wish $F_{\mu\nu}$ to transform. However, it is not yet a satisfactory field because it is a differential operator. Show that $[D_\mu, D_\nu] = D_\mu D_\nu - D_\nu D_\mu$ is a matrix valued field and evaluate it as a function of A and ∂A.

c) Assume the gauge group is a Lie group generated by matrices t_a satisfying the Lie algebra

$$[t_a, t_b] = if_{abc}t_c \tag{2}$$

where f_{abc} are called the structure constants of the group. Then we can expand the gauge potential $A_\mu(x) = \sum_a t_a A^a_\mu$. Defining $F^a_{\mu\nu}$ by $[D_\mu, D_\nu] = -ig\sum_a t_a F^a_{\mu\nu}$, express $F^a_{\mu\nu}$ in terms of A^a_μ, its derivatives, and the structure constants.

42. Show without using perturbation theory that the function

$$\Delta_A(x,y) \equiv \frac{\langle out|T[\phi(x)\phi^\dagger(y)]|in\rangle}{\langle out|in\rangle}$$

for a charged scalar field ϕ in an external electromagnetic field, is a Green's function for the differential operator $m^2 - (\partial - iqA)^2$:

$$[m^2 - (\partial - iqA)^2]\Delta_A(x,y) = -i\delta(x-y).$$

You will need to use the canonical commutation rules for the charged scalar field and the fact that ϕ satisfies the equation $[m^2 - (\partial - iqA)^2]\phi = 0$.

43. **Positron scattering**

a) Write down an explicit formula for

$$\frac{\langle out|d^{out}_{\lambda'}(\boldsymbol{q}')d^{in\dagger}_{\lambda}(\boldsymbol{q})|in\rangle}{\langle out|in\rangle}$$

as an expansion in powers of $e\tilde{A}_\mu(p)$, where the states are positron states.

b) The terms in the expansion of part a) are spinor matrix elements involving $v, \bar{v}$ rather than $u, \bar{u}$ as electron scattering would. By using the relation of v to u, $v_\lambda(p) = i\gamma^2\gamma^0\overline{u}_\lambda(p)^T$ and $\overline{v}_\lambda(p) = u^T_\lambda(p)i\gamma^2\gamma^0$ and $(i\gamma^2\gamma^0)\gamma^\mu = -\gamma^{\mu T}(i\gamma^2\gamma^0)$, show explicitly that the positron amplitude in a field A_μ is the same as the electron amplitude in the field $-A_\mu$. [Of course this follows immediately from the charge conjugation properties of the theory, but I want you to follow how it happens in the Feynman diagram amplitudes. Take care with the overall sign.] What is the relation between the differential cross sections for positron and electron scattering in the same field A_μ in Born approximation?