Quantum Field Theory

Problem Set 12

Due: 12 April 2019

44. Consider the charged scalar field in the presence of an external electromagnetic field. The Hamiltonian is given in problem 8 (of Set 2), and in problem 42 (of Set 11) you showed that

$$\Delta_A(x,y) \equiv \frac{\langle out|T[\phi(x)\phi^\dagger(y)]|in\rangle}{\langle out|in\rangle}$$

is a Green's function for the differential operator $m^2 - (\partial - iqA)^2$:

$$[m^2 - (\partial - iqA)^2]\Delta_A(x,y) = -i\delta(x-y).$$

a) Show that under $A \to A + \delta A$, $\ln\langle out|in\rangle_A$ changes (to first order in δA) by the amount

$$\delta \ln\langle out|in\rangle_A = i\int d^4x \delta A^\mu(x)\frac{\langle out|j_\mu(x)|in\rangle_A}{\langle out|in\rangle_A},$$

where $j_\mu = -iq(\phi^\dagger\partial_\mu\phi - \phi\partial_\mu\phi^\dagger) - 2q^2A_\mu\phi^\dagger\phi$. [Hint: first obtain the analogue, for the charged scalar field, of Eq.(7.20) in the lecture notes.]

b) Provisionally make the identification

$$\frac{\langle out|j^\mu(x)|in\rangle_A}{\langle out|in\rangle_A} = \lim_{y\to x}\Bigg\{-iq(\partial_x^\mu - iqA(x))\frac{\langle out|T[\phi(x)\phi^\dagger(y)]|in\rangle_A}{\langle out|in\rangle_A}$$
$$-iq\frac{\langle out|T[\phi(x)\phi^\dagger(y)]|in\rangle_A}{\langle out|in\rangle_A}(-\overleftarrow{\partial}_y^\mu - iqA(y))\Bigg\}, \qquad (1)$$

which is formally valid. Then from part a), show that

$$\langle out|in\rangle_A = \frac{C}{\det[m^2 - (\partial - iqA)^2]}.$$

c) Confirm the result of part b) to second order in A by using time dependent perturbation theory (*i.e.* by expanding the Dyson formula in interaction picture).

45. In the vacuum polarization calculation we used the Feynman trick

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(Ax + B(1-x))^2} \qquad (2)$$

to combine propagator denominators.

a) Prove this formula by direct integration.

b) This is a special case of a formula combining n denominators

$$\frac{1}{A_1 A_2 \cdots A_n} = (n-1)! \int_0^1 dx_1 \cdots dx_n \delta(1 - \sum_k x_k) \frac{1}{[\sum_k x_k A_k]^n} \tag{3}$$

Prove this generalization. A good first step is to write $1/A_k = \int_0^\infty dT_k e^{-T_k A_k}$. then change variables to $T = \sum_k T_k$ and $x_k = T_k/T$. Then do the integral over T.

46. **Integration over Euclidean D dimensional spacetime.**

a) Confirm the validity of the replacement shown in Eq (10.28) of the lecture notes.

b) Derive equation (10.32) in our lecture notes, following the suggestions in the text.

c) Derive equation (10.33), using the integral representation for the Euler beta function

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 dt t^{x-1}(1-t)^{y-1}. \tag{4}$$

You will have to scale out A and do an appropriate change of integration variables in polar coordinates after the angular integration.

47. Instead of using the cutoff Λ to do the divergent integrals, equations (10.35)and (10.37), use the results proved in the previous problem to do the integrals

$$\int \frac{d^D p}{(2\pi)^D} \frac{1}{[p^2 + C^2]^2}, \qquad \int \frac{d^D p}{(2\pi)^D} \frac{p^2}{[p^2 + C^2]^2} \tag{5}$$

for general Euclidean spacetime dimension D. After the angular integrals, the integrals over the magnitude p will converge for sufficiently small $D < 4$. Now consider your results in the limit $D \to 4$, and compare the result to that using the cutoff Λ described in the notes. You will find poles at $D = 4$ instead of $\ln \Lambda$'s.