Quantum Field Theory

Problem Set 13

Due: 24 April 2019 (Last day of class)

Reading:] Lecture Notes Chapters 11-13.

48. **Vacuum Polarization contribution to the Lamb shift**. We derived the relation between D and E, which after charge renormalization, to first order in e^2 is

$$\tilde{E}(k) = \left[1 + \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left\{1 + x(1-x)\frac{k^2}{m^2}\right\}\right]\tilde{D}(k)$$

In consequence, the potential energy of the electron in a hydrogen atom is modified from the lowest order $-e^2/4\pi|\mathbf{x}|$. In this problem you are to calculate the effect of this modification on the hydrogen energy levels, by inserting the change of the potential into lowest order non-relativistic perturbation theory.

a) Explain why values of $k^2/m^2 \ll 1$ as well as $k^{02} \ll \vec{k}^2$ are involved in this calculation.

b) The estimates of part a) justify expanding the logarithm in the formula to estimate the integral. Show that the extra potential energy is of the form $-V_1\delta^3(\vec{x})$ and find the constant V_1.

c) Explain why only s levels ($l = 0$) will be affected by this perturbation.

d) Calculate the shift in the $2s$ state in electron-volts. (Remember that we have put $\hbar = c = 1$. Restore them at some late stage to get dimensions right.) Note that this is a relatively small contribution to the Lamb shift: it is an order of magnitude smaller than and has the opposite sign to the main contribution.

49. <u>Path integral expression for the Partition function</u>
It is sometimes interesting to interpret the imaginary time path integral directly as giving the statistical mechanics of the system at non-zero temperature. The partition function $Z(\beta)$ at temperature β^{-1} is defined as $Tr\ e^{-\beta H} = \int dq \langle q|e^{-\beta H}|q\rangle$, where q is a complete set of coordinates. But we have an expression for $\langle q''|e^{-\beta H}|q'\rangle$ as an integral over all paths from q' at $t = 0$ to q'' at $t = -i\beta$. So we get $Z(\beta) = \int Dq \exp\{i\int_0^\beta L(-i\tau)(-i)d\tau\}$ where the integral is over all paths with $q(-i\beta) = q(0)$. For the case of one degree of freedom with $L = \frac{1}{2}m\dot{q}^2 - V(q)$, this becomes

$$Z(\beta) = \int Dq \exp\left[-\int_0^\beta \left\{\frac{1}{2}m\left(\frac{dq}{d\tau}\right)^2 + V(q)\right\}d\tau\right], \qquad q(\beta) = q(0)$$

To define the integral, we break the interval 0 to β into N parts of length ϵ and replace $\int (dq/d\tau)^2 \, d\tau$ by $(1/\epsilon)\sum_{r=1}^{N}(q_{r+1}-q_r)^2$ (with $q_{N+1} \equiv q_1$), $\int V(q)d\tau$ by $\epsilon \sum_{r=1}^{N} V(q_r)$, and then evaluate $(m/2\pi\epsilon)^{N/2} \int dq_1 \dots dq_N$. In practice we usually calculate ratios of path integrals or fix normalizations using closure, thus avoiding an explicit evaluation of the path integral. In this exercise we shall calculate $Z(\beta)$ directly for the harmonic oscillator, $V(q) = \frac{1}{2}m\omega^2 q^2$.

a) Show that the formula for Z takes the form

$$Z(\beta) = \left(\frac{m}{2\pi\epsilon}\right)^{N/2} \int \prod_{r=1}^{N} dq_r \exp\left[-\frac{1}{2}\sum_{r,s=1}^{N} A_{rs} q_r q_s\right].$$

finding an explicit expression for the $N \times N$ matrix A_{rs}.

b) We can always change variables to linear combinations of q_r which makes A_{rs} diagonal. Show that $Z(\beta) = (m/\epsilon)^{N/2} \prod_{\nu=1}^{N} \lambda_\nu^{-1/2}$ where λ_ν are the eigenvalues of the equations $\sum_s A_{rs} q_s = \lambda q_r$. (In fact $\prod \lambda_\nu = \det A$.)

c) Show that $\lambda_\nu = 2(m/\epsilon)(1 - \cos\theta_\nu) + m\epsilon\omega^2$, $\theta_\nu = 2\nu\pi/N$, $\nu = 0, \dots, N-1$. (Remember $q_{N+1} \equiv q_1$.) This is essentially the same mathematical problem as finding the normal modes of a chain of identical masses connected by identical springs. Remember your classical mechanics!

d) Evaluate $Z(\beta)$ using the identity $2(\cos N\theta - 1) = \prod_{\nu=0}^{N-1}(2\cos\theta - 2\cos\theta_\nu)$ (which is true because $\cos N\theta$ is a polynomial in $\cos\theta$ and the R.H.S. has the same zeros as the L.H.S. and the right coefficient of $\cos^N\theta$). Verify that it agrees with a direct evaluation of $\mathrm{Tr}e^{-\beta H}$ using standard raising and lowering operators $a, a^\dagger$.

Note that we could interpret this calculation as the partition function of an elastic string of length β in a potential V in classical statistical

50. **Compton Scattering**: In this problem and the next (Problem 51), you will calculate the differential cross section for a photon scattering of an electron at rest. In this problem you are to derive the squared amplitude. According to our discussion of the free quantum field for the photon, its polarization is specified by a complex three vector, $\boldsymbol{\epsilon}$ perpendicular to its momentum $\boldsymbol{k}$. it is convenient to denote this as a four vector ϵ_μ with time component $\epsilon_0 = 0$. Then it follows that an incoming (outgoing) photon corresponds to a factor of ϵ (ϵ^*) dotted into the vertex it enters.(leaves).

a) Write down the Feynman amplitude for this process in terms of Dirac spinors for the initial and final electrons and the polarization vectors $\epsilon_\mu(k)$ and $\epsilon'_\mu(k')$ for the initial and final photons. Show that the change $\epsilon_\mu \to \epsilon_\mu + Ck_\mu$ leaves the amplitude unchanged. This verifies gauge invariance, and provides the flexibility to use a polarization vector with $\epsilon_0 \neq 0$, provided $k^\mu\epsilon_\mu = 0$.

b) Express the squared amplitude summed over final electron spins and averaged over initial electron spins as traces of products of gamma matrices.

c) Sum over final and average over initial photon polarizations, using the results for the **spatial** components. (The time components are zero!).

$$\sum_{pol} \epsilon_l \epsilon_m^* = \delta_{lm} - \frac{k_l k_m}{\boldsymbol{k}^2}, \tag{1}$$

and similarly for ϵ'.

d) Evaluate the resulting traces and confirm that the result matches the one quoted in Srednicki equation (11.50).

51. S, Problem 11.2. This exercise guides you through the cross section evaluation for Compton scattering starting from the result obtained is part d) of the previous problem.