

# LECTURE NOTES FOR QUANTUM FIELD THEORY

by

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## 1. Introduction

One of the most basic facts about relativistic physics is the equivalence of mass and energy, as in Einstein's famous relation  $E = mc^2$ . This fact has profound consequences for relativistic quantum mechanics, because of Heisenberg's uncertainty principle

$$\Delta x \Delta p \geq \hbar. \tag{1.1}$$

In ordinary nonrelativistic quantum mechanics this follows from the fact that the position and momentum are represented by operators that do not commute:  $[x, p] = i\hbar$ . According to (1.1) an accurate position measurement can be made only at the expense of a compensatingly large inaccuracy in the momentum of the particle. This implies a correspondingly large expense of energy in the accurate measurement of position. In nonrelativistic quantum mechanics the rest energy of a massive particle is effectively infinite so one can subject a particle to arbitrarily large energy changes without compromising its integrity as a particle, and accurate position measurements are compatible with particle number conservation. The concept of an  $N$  particle wave function as a function of the  $N$  particle coordinates is sustainable. This is drastically changed in relativistic quantum mechanics. As soon as one attempts to measure the position of a particle to an accuracy much less than the particle's Compton wavelength,  $\hbar/mc$ , the associated momentum uncertainty is  $> mc$ , corresponding to an energy change larger than  $mc^2$ . Thus accurate position measurements require the supply of energy sufficient to produce additional particles. If the particle possesses a conserved charge, additional such particles can be produced but only in association with particles of opposite charge. Thus we should expect any consistent version of relativistic quantum mechanics to require abandoning the concept of a wave function for a system with a definite number of particles. Further, in retrospect, it is not surprising that when there are conserved charges, relativistic quantum physics requires antiparticles.

The preceding is not meant to be a rigorous argument, but rather a simple explanation of why we must expect at least the complexities of quantum field theory when we attempt to extend quantum mechanics into the relativistic domain. The many-body aspects of relativistic quantum mechanics are universal, but the manner in which they make their appearance varies. In quantum electrodynamics it is immediate once one tries to interpret classical radiation in terms of photons. For massive particles, like the electron, it arises from the existence of negative energy solutions of the Dirac equation.

The first semester of quantum field theory will focus primarily on quantum electrodynamics, the theory of electrons and positrons interacting with the quantized electromagnetic field. We shall begin however with a much simpler quantum field theory— that of a self interacting scalar field. Then we shall show how Dirac's theory of electrons and positrons can be converted to a quantum field theory through the device of "second quantization". We shall study this Dirac quantum field theory first in the presence of external (classical) electromagnetic fields. Only then will we turn to the problem of quantizing the electromagnetic field and studying it in interaction with electrons and positrons.

## 2. Lorentz Invariance

In elementary discussions of special relativity we learn that frames of reference with a constant relative velocity  $V$  have their coordinates related by the Lorentz “boost”

$$x' = \gamma(x + Vt), y' = y, z' = z, t' = \gamma(t + Vx)$$

where  $\gamma = 1/\sqrt{1 - V^2}$ . But more generally the boost could be in any direction specified by a three vector  $\vec{V}$ . In addition to boost invariance, we also require rotational invariance, a symmetry we are very familiar with in nonrelativistic mechanics. We define a general Lorentz transformation of space-time coordinates  $x^\mu = (t, x, y, z) \equiv (x^0, x^1, x^2, x^3)$  as a linear transformation

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$$

where  $\Lambda$  preserves Minkowski scalar products  $v \cdot w = v^1 w^1 + v^2 w^2 + v^3 w^3 - v^0 w^0 \equiv v^\mu v^\nu \eta_{\mu\nu}$ . ( $\eta_{11} = \eta_{22} = \eta_{33} = -\eta_{00} = 1$ ,  $\eta_{\mu\nu} = 0$  for  $\mu \neq \nu$ ). Note that we use the convention that repeated indices are summed. This requirement implies the following constraints on  $\Lambda$ :

$$\eta_{\rho\sigma} \Lambda^\rho_\mu \Lambda^\sigma_\nu = \eta_{\mu\nu}.$$

You should convince yourself that the special boost in the  $x$  direction does indeed preserve Minkowski scalar products. The Poincaré group consists of Lorentz transformations together with translations

$$x^\mu \rightarrow \Lambda^\mu_\nu x^\nu + a^\mu.$$

According to special relativity the laws of physics should look the same in all frames related by Lorentz transformations. The systematic way to implement this requirement is to identify all physical quantities as the components of 4 tensors  $A^{\mu\nu\dots}$  which transform as

$$A'^{\mu\nu\dots} = \Lambda^\mu_\rho \Lambda^\nu_\sigma \dots A^{\rho\sigma\dots}$$

For example energy and momentum are the components of a four vector  $p^\mu = (E, \vec{p})$  which transform just like the coordinates  $x^\mu$ .

$$p^\mu \rightarrow p'^\mu = \Lambda^\mu_\nu p^\nu$$

Tensors written with index superscripts are called contravariant tensors. It is also useful to introduce covariant tensors  $A_{\mu\nu\dots}$ , written with index subscripts. A covariant index transforms like  $\partial/\partial x^\mu$ :

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial}{\partial x^\rho} = \frac{\partial}{\partial x^\rho} (\Lambda^{-1})^\rho_\mu = \frac{\partial}{\partial x^\rho} \eta^{\rho\kappa} \Lambda^\sigma_\kappa \eta_{\sigma\mu} \equiv \Lambda_\mu^\rho \frac{\partial}{\partial x^\rho}$$

A general tensor can have any number of upper and lower indices. the metric tensor is used to raise and lower indices.

The  $\Lambda$ 's can be divided into 4 disjoint sets according to the signs of  $\det \Lambda$  and  $\Lambda^0_0$ . This is because it is easy to show from the above property that  $(\det \Lambda)^2 = 1$  and  $(\Lambda^0_0)^2 \geq 1$ . Thus a continuous variation of  $\Lambda$  always stays within one of these sets. In the following we restrict ourselves to the *proper* Lorentz Group, *i.e.* with  $\det \Lambda = +1$  and  $\Lambda^0_0 \geq +1$ . The complete Lorentz group is then obtained by adjoining parity and time reversal.

Lorentz transformations with  $\Lambda^0_k = \Lambda^k_0 = 0$  are simply rotations and form a subgroup. We know from basic quantum mechanics all the unitary irreducible representations of the Rotation group, namely those labeled by angular momentum  $j = 0, 1/2, 1, 3/2, \dots$ . This conclusion is reached by considering infinitesimal rotations which are generated by the angular momentum  $\vec{J}$ ,  $U(R) = e^{-i\theta\hat{u}\cdot\vec{J}} \approx 1 - i\theta\hat{u}\cdot\vec{J}$ . The commutators of  $J^k$  are well-known:

$$[J^k, J^l] = i\hbar\epsilon_{klm}J^m$$

and the  $|jm\rangle$  basis of eigenstates of  $\vec{J}^2$ ,  $J_3$  is constructed in the familiar way. The unitary representations of the Lorentz group must be extensions of these,  $U(\Lambda) = U(R)$  when  $\Lambda^0_k = \Lambda^k_0 = 0$ . To find the generators of boosts, consider the boost in the  $x$  direction of the momentum of a particle of mass  $m$ :

$$p^1 = \gamma(p^1 + V\sqrt{\vec{p}^2 + m^2}) \approx p^1 + V\sqrt{\vec{p}^2 + m^2}$$

for infinitesimal  $V$ . This should be identified with  $p^1 - iV[p^1, K^1]$ , from which we infer

$$K^1 = -(x^1\sqrt{\vec{p}^2 + m^2} + \sqrt{\vec{p}^2 + m^2}x^1)/2 + F(\vec{p})$$

. The symmetrized product in the first term is to keep  $K$  hermitian. To find  $F$  we consider the transform of the coordinate

$$x^1(t') \approx x^1(t) + Vt \approx x^1(t) - V(x^1\dot{x}^1(t) + \dot{x}^1(t)x^1)/2 + Vt$$

But

$$-iV[x^1, K^1] = -\frac{1}{2}\left(x^1\frac{p^1}{\sqrt{\vec{p}^2 + m^2}} + \frac{p^1}{\sqrt{\vec{p}^2 + m^2}}x^1\right) + \frac{\partial F}{\partial p^1} = -\frac{1}{2}(x^1\dot{x}^1 + \dot{x}^1x^1) + \frac{\partial F}{\partial p^1}$$

Thus we are led to

$$K^1 = -(x^1\sqrt{\vec{p}^2 + m^2} + \sqrt{\vec{p}^2 + m^2}x^1)/2 + p^1t$$

We easily see that  $K^1$  is constant in time. Boosts in all three directions are generated by

$$K^k = -(x^k\sqrt{\vec{p}^2 + m^2} + \sqrt{\vec{p}^2 + m^2}x^k)/2 + p^kt$$

It is now a straightforward exercise to complete the commutator algebra of Lorentz generators:

$$[J^k, J^l] = i\hbar\epsilon_{klm}J^m, \quad [K^k, J^l] = i\hbar\epsilon_{klm}K^m, \quad [K^k, K^l] = -i\hbar\epsilon_{klm}J^m$$

An alternative more covariant presentation of the Lorentz algebra is to define  $M_{ij} = \epsilon_{ijk}J^k$ ,  $M_{0i} = K^i$ , and find

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho})$$

where now and henceforth we will choose units so that  $\hbar = c = 1$ .

In classical electromagnetic theory we learn that Maxwell's equations are covariant under Lorentz transformations of the electromagnetic field:

$$F'^{\mu\nu}(x') = \Lambda^\mu_\rho\Lambda^\nu_\sigma F^{\rho\sigma}(x) = \Lambda^\mu_\rho\Lambda^\nu_\sigma F^{\rho\sigma}(\Lambda^{-1}x')$$

Since  $F^{\mu\nu}(x)$  is a field with a different value at each point, notice that two changes are going on simultaneously: the components of  $F$  are mixed *and* the old and new fields are compared at different space-time

coordinates  $x'$ ,  $x$  related by  $x' = \Lambda x$ . We are going to consider at the beginning a scalar field  $\phi(x)$  with only one component. The scalar field then has the much simpler Lorentz transformation

$$\phi'(x') = \phi(\lambda^{-1}x')$$

The simplest Lorentz covariant scalar field equation is the Klein-Gordon equation

$$-\eta^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} \phi + \mu^2 \phi = 0$$

To show Lorentz covariance we want to prove

$$-\eta^{\mu\nu} \frac{\partial^2}{\partial x'^\mu \partial x'^\nu} \phi(\Lambda^{-1}x') + \mu^2 \phi(\Lambda^{-1}x') = 0.$$

this follows through the chain rule

$$\eta^{\mu\nu} \frac{\partial^2}{\partial x'^\mu \partial x'^\nu} = \eta^{\mu\nu} (\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu \frac{\partial^2}{\partial x^\rho \partial x^\sigma} = \eta^{\rho\sigma} \frac{\partial^2}{\partial x^\rho \partial x^\sigma}$$

### 3. Scalar Quantum Field Theory

#### 3.1. THE FREE SCALAR FIELD

The field concept goes all the way back to Faraday in the first part of the nineteenth century. It was of course indispensable to a proper understanding of electromagnetism which he pioneered. This first field theory is actually quite complex involving three components for each of the electric and magnetic fields. For our first look at quantum field theory we consider the much simpler case of the single component scalar field.

A field is a dynamical variable  $\phi(\mathbf{x}, t)$  assigned to each point of space. One is therefore positing from the beginning an infinite number of degrees of freedom. The simplest relativistic equation of motion for such a field is a linear wave equation (henceforth we shall use units in which the speed of light  $c = 1$ ):

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + \mu^2\right)\phi \equiv (-\partial^2 + \mu^2)\phi = 0.$$

The parameter  $\mu$  clearly has dimensions of 1/Length, and represents an inverse wavelength (or wave number) rather than a mass. After quantization, we shall see that the particles associated with the quantum field have mass  $\hbar\mu$ . We are of course very familiar with the nature of the solutions to these equations: general superpositions of plane waves, called wave packets:

$$\phi(\mathbf{x}, t) = \int d^3k f(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega(\mathbf{k})t} \approx \int d^3k f(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega(\mathbf{k}_0)t - i(\mathbf{k} - \mathbf{k}_0)\cdot\nabla\mathbf{k}\omega|_{\mathbf{k}_0}} \approx e^{-i\omega(\mathbf{k}_0)t + i\mathbf{k}_0\cdot\mathbf{v}_g t} \phi(\mathbf{x} - \mathbf{v}_g t, 0)$$

where  $\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2 + \mu^2}$ , moving with group velocity  $v_g = \mathbf{k}/\sqrt{\mathbf{k}^2 + \mu^2}$ .

Notice that the wave equation is invariant in form under Lorentz transformations  $\phi'(x') = \phi(x)$  where  $x'^\mu = \Lambda^\mu{}_\nu x^\nu$  and  $\Lambda$  is a Lorentz transformation satisfying  $\eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = \eta_{\mu\nu}$ . This condition on  $\Lambda$  is just the requirement that scalar products of four vectors  $v \cdot w \equiv \eta_{\mu\nu} v^\mu w^\nu = \vec{v} \cdot \vec{w} - v^0 w^0$  are invariant. One easily can see that  $\partial/\partial x_\mu$  transforms as a four-vector so that  $\partial^2 = \partial^\mu \partial^\nu \eta_{\mu\nu}$  is invariant.

To discuss the quantum mechanics of this field we must identify the canonical variables of the system. Since the equation of motion is of second order we can easily guess that the momentum conjugate to  $\phi(\mathbf{x})$  is just  $\pi(\mathbf{x}) = \dot{\phi}$ . A more systematic approach is to find an action principle which implies the field equation. Recall that for dynamical systems with discretely labeled degrees of freedom  $q_i(t)$  one forms the action from the Lagrangian  $L(q(t), \dot{q}(t), t)$  via  $S = \int_{t_1}^{t_2} dt L$ . Hamilton's principle states that the equations of motion follow from the requirement that  $S$  is stationary under variations  $\delta q_i(t)$  with  $\delta q_i(t_1) = \delta q_i(t_2) = 0$ . By definition the momentum conjugate to  $q_i$  is  $p_i \equiv \partial L / \partial \dot{q}_i$ . For our field the spatial coordinates  $\mathbf{x}$  play the role of the index  $i$ , and the Lagrangian will be an integral over spatial coordinates. It is easy to see that Hamilton's principle applied to the action

$$S = \int_{t_1}^{t_2} dt \int d^3x \frac{1}{2} (\dot{\phi}^2 - (\nabla\phi)^2 - \mu^2 \phi^2) \equiv \int d^4x \mathcal{L}$$

$$\mathcal{L} = -\frac{1}{2} [\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \mu^2 \phi^2]$$

implies the Klein-Gordon equation. The second form shows that the Lagrange density  $\mathcal{L}$  is a Lorentz covariant scalar field, so its integral is a Lorentz invariant.

From this action, it is evident that the above guess for  $\pi \equiv \partial \mathcal{L} / \partial \dot{\phi} = \dot{\phi}$  is correct. The canonical Hamiltonian is

$$H \equiv \int d^3x [\dot{\phi} \pi - \mathcal{L}] = \int d^3x \frac{1}{2} (\pi^2 + (\nabla\phi)^2 + \mu^2 \phi^2).$$

One can easily check that Hamilton's equations obtained from this Hamiltonian give that same old wave equation. To quantize canonically, we promote  $\phi, \pi$  to operators with the canonical commutation relations

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\hbar \delta(\mathbf{x} - \mathbf{y}).$$

In the language of Hamiltonian mechanics with canonical variables  $q_i, p_i$ ,  $\phi$  is analogous to  $q$  and  $\pi$  to  $p$ , and the spatial coordinate to the index  $i$  that distinguishes independent degrees of freedom.

To find the energy eigenstates, we note the close resemblance of our Hamiltonian to that of a system of **coupled harmonic oscillators**. The only thing a little strange is that the oscillator coordinates are labeled by a continuous index. This can be remedied by replacing space by a lattice  $a\mathbf{n}$  of spacing  $a$  which we send to zero after finding the eigenstates. Thus we replace  $\phi(a\mathbf{n})$  with  $\phi_{\mathbf{n}}$  and  $\pi(a\mathbf{n})$  by  $\pi_{\mathbf{n}}/a^3$ . We divide by  $a^3$  so that  $[\phi_{\mathbf{n}}, \pi_{\mathbf{m}}] = i\hbar \delta_{\mathbf{n},\mathbf{m}}$ . Doing this the Hamiltonian becomes

$$H_{lattice} = \frac{a^3}{2} \sum_{\mathbf{n}} \left( \frac{1}{a^6} \pi_{\mathbf{n}}^2 + \mu^2 \phi_{\mathbf{n}}^2 + \frac{1}{a^2} \sum_{\hat{i}} (\phi_{\mathbf{n}+\hat{i}} - \phi_{\mathbf{n}})^2 \right).$$

On the lattice our system is a coupled system of oscillators. To solve it, we just have to find the normal modes. This is done by a change of variables which renders diagonal the coupling matrix

$$V_{\mathbf{n},\mathbf{m}} = (\mu^2 + \frac{2d}{a^2}) \delta_{\mathbf{n},\mathbf{m}} - \frac{1}{a^2} \sum_{\hat{i}} (\delta_{\mathbf{n}+\hat{i},\mathbf{m}} + \delta_{\mathbf{n},\mathbf{m}+\hat{i}}),$$

*i.e.* we want to find the eigenvectors and eigenvalues of  $V$ . It is easy to see that the eigenvectors are

$u_{\mathbf{m}} = e^{i\mathbf{K}\cdot\mathbf{m}}$ ,  $-\pi < K_i < \pi$ , belonging to eigenvalue

$$\lambda \equiv \omega^2 = \mu^2 + \frac{2}{a^2} \left( d - \sum_i \cos \mathbf{K} \cdot \hat{\mathbf{i}} \right).$$

Thus the transformation to normal modes is given by

$$\begin{aligned} \phi_{\mathbf{n}} &= \int_{-\pi}^{\pi} \frac{d^3 K}{(2\pi)^{3/2}} e^{i\mathbf{K}\cdot\mathbf{n}} Q(\mathbf{K}) \\ \pi_{\mathbf{n}} &= \int_{-\pi}^{\pi} \frac{d^3 K}{(2\pi)^{3/2}} e^{i\mathbf{K}\cdot\mathbf{n}} P(\mathbf{K}) \end{aligned}$$

or inversely by

$$\begin{aligned} Q(\mathbf{K}) &= \sum_{\mathbf{n}} \frac{1}{(2\pi)^{3/2}} e^{-i\mathbf{K}\cdot\mathbf{n}} \phi_{\mathbf{n}} \\ P(\mathbf{K}) &= \sum_{\mathbf{n}} \frac{1}{(2\pi)^{3/2}} e^{-i\mathbf{K}\cdot\mathbf{n}} \pi_{\mathbf{n}}. \end{aligned}$$

The inverse equations directly imply the commutation relations

$$[Q(\mathbf{K}), P(\mathbf{L})] = i\hbar\delta(\mathbf{K} + \mathbf{L}).$$

Note also that if  $\phi$  and  $\pi$  are hermitian, as we assume, then we have  $Q(\mathbf{K})^\dagger = Q(-\mathbf{K})$  and  $P(\mathbf{K})^\dagger = P(-\mathbf{K})$ .

Expressed in terms of normal modes the lattice Hamiltonian becomes

$$H_{lattice} = \int_{-\pi}^{\pi} d^3 K \left( \frac{1}{2a^3} P(\mathbf{K})P(-\mathbf{K}) + \frac{a^3}{2} \omega^2(\mathbf{K}) Q(\mathbf{K})Q(-\mathbf{K}) \right).$$

In the standard fashion raising and lowering operators can be constructed in terms of which

$$\begin{aligned} Q(\mathbf{K}) &= \sqrt{\frac{\hbar}{2\omega(\mathbf{K})a^3}} (A(\mathbf{K}) + A^\dagger(-\mathbf{K})) \\ P(\mathbf{K}) &= -i\sqrt{\frac{\hbar\omega(\mathbf{K})a^3}{2}} (A(\mathbf{K}) - A^\dagger(-\mathbf{K})) \end{aligned}$$

with  $[A(\mathbf{K}), A^\dagger(\mathbf{L})] = \delta(\mathbf{K} - \mathbf{L})$ , and the Hamiltonian becomes the familiar

$$H_{lattice} = \int_{-\pi}^{\pi} d^3 K \hbar\omega(\mathbf{K}) \frac{1}{2} (A(\mathbf{K})A^\dagger(\mathbf{K}) + A^\dagger(\mathbf{K})A(\mathbf{K})).$$

$A^\dagger$  and  $A$  are of course eigenoperators of  $H_{Lattice}$  with eigenvalues  $\pm\omega(\mathbf{K})$ .

Now we can describe the exact energy eigenstates of the system. The ground state (the vacuum)  $|0\rangle$  is annihilated by all the  $A$ 's:  $A(\mathbf{K})|0\rangle = 0$ , and its energy  $E_0 = (\hbar/2)\delta(\mathbf{0}) \int d^3 K \omega(\mathbf{K})$ . We shall measure all

energies relative to  $E_0$ , *i.e.* from now on we take  $E_0 = 0$ . This means our energy operator is redefined to be

$$H - E_0 = \int_{-\pi}^{\pi} d^3 K \hbar \omega(\mathbf{K}) A^\dagger(\mathbf{K}) A(\mathbf{K})$$

The excited states are obtained by applying any number of  $A^\dagger$ 's to the vacuum. The simplest one is  $A^\dagger(\mathbf{K}) |0\rangle$  with energy

$$E(\mathbf{K}) = \hbar \sqrt{\mu^2 + \frac{2}{a^2} \left( d - \sum_{\hat{i}} \cos \mathbf{K} \cdot \hat{i} \right)}.$$

At this point we can consider the continuum limit  $a \rightarrow 0$ . All excited states with finite  $\mathbf{K}$  will have infinite energy and be dynamically irrelevant. However since the components of  $\mathbf{K}$  are continuous, we may consider states with  $\mathbf{K} = a\mathbf{k}$ . The energy of these states becomes  $\hbar \sqrt{\mu^2 + k^2}$  independent of  $a$  in the continuum limit. We identify this energy as that of a relativistic particle of mass  $\hbar\mu$  and momentum  $\hbar\mathbf{k}$ .

We can pass to the continuum limit in all of our results by changing variables from  $\mathbf{K}$  to  $\mathbf{k}$  whose components range from  $-\pi/a$  to  $\pi/a$  which become  $-\infty$  to  $\infty$  as  $a \rightarrow 0$ . At the same time we replace  $A(\mathbf{K}) \equiv a(\mathbf{k})/a^{3/2}$  so that  $[a(\mathbf{k}), a^\dagger(\mathbf{q})] = \delta(\mathbf{k} - \mathbf{q})$ . After these changes we find as  $a \rightarrow 0$ :

$$\begin{aligned} \phi_{\mathbf{n}} &\rightarrow \phi(\mathbf{x}) = \int \frac{d^3 k}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\omega(\mathbf{k})}} (a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}) \\ \frac{\pi_{\mathbf{n}}}{a^3} &\rightarrow \pi(\mathbf{x}) = -i \int \frac{d^3 k}{(2\pi)^{3/2}} \sqrt{\frac{\hbar\omega(\mathbf{k})}{2}} (a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}) \\ H - E_0 &\rightarrow H_\phi - E_0 = \int d^3 k \hbar \omega(\mathbf{k}) a^\dagger(\mathbf{k}) a(\mathbf{k}) \end{aligned}$$

where now  $\omega(\mathbf{k}) = \sqrt{\mu^2 + \mathbf{k}^2}$ . It should of course be clear that we really never needed the lattice in obtaining these results: we simply had to substitute the above expansions for  $\phi, \pi$  into  $H_\phi$ . The lattice only served to make absolutely concrete our assertion that the field system was a set of coupled oscillators.

We should also note that the momentum operator of the continuum field theory, identified as the generator of translations, is

$$\mathbf{P} = - \int d^3 x \pi(\mathbf{x}) \nabla \phi(\mathbf{x}) = \int d^3 k \hbar \mathbf{k} a^\dagger(\mathbf{k}) a(\mathbf{k})$$

confirming the interpretation of  $\hbar\mathbf{k}$  as the momentum of the one particle state  $a^\dagger(\mathbf{k}) |0\rangle$ .

If we apply two  $a^\dagger$ 's to the vacuum we get a two particle state

$$a^\dagger(\mathbf{k}_1) a^\dagger(\mathbf{k}_2) |0\rangle$$

with total momentum  $\hbar(\mathbf{k}_1 + \mathbf{k}_2)$  and total energy  $\hbar(\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2))$ . Clearly this energetics is that of noninteracting particles. It is highly significant that all multiparticle states are completely symmetric under interchange of the labels of any pair of particles: the scalar field theory predicts Bose statistics for the associated particles!

The example of the scalar field illustrates the main physical aspects of quantum field theory

1. It predicts multi-particle states together with their statistics (Bose or Fermi).
2. It incorporates the requirements of Special Relativity (Poincaré invariance).

3. One can consider two classical limits ( $\hbar \rightarrow 0$ ):
  - (a) With  $\mu$  fixed the limit gives a classical field theory. In this case the Compton wavelength  $1/\mu$  is finite in the classical limit.
  - (b) With  $m = \hbar\mu$  fixed it gives a classical theory of point particles (the Compton wavelength vanishes).
  - (c) Thus quantum field theory unites the particle and field concepts.
4. Quantum field theories with linear field equations predict noninteracting (free) particles. Interactions will arise if the field equations have nonlinear terms. Such terms are associated with terms in the action or Hamiltonian with three or more powers of fields. The presence of such terms implies that particle number is not conserved.
5. The scalar field describes spinless particles; more general fields *e.g.* spinor, vector, tensor describe particles of higher spin.

Finally, let us consider how an approximate quantum particle interpretation can be retrieved from this quantum field theory. Remember the observables are quantum fields, not particle coordinates! To discover properties of a single particle wave packet,

$$|f\rangle \equiv \int d^3k f(\mathbf{k}) a^\dagger(\mathbf{k}) |0\rangle,$$

we must consider what we can get from measurements of the quantum field

$$\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\omega(\mathbf{k})}} (a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} + a^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega t}).$$

First it is easily shown that  $\langle f | \phi(\mathbf{x}, t) | f \rangle = 0$ . That is, if we make many measurements of  $\phi$ , we find zero on average. This means nothing more than  $\phi$  is negative as often as it is positive. A more sophisticated measurement is to measure  $\phi(\mathbf{x}, t)\phi(\mathbf{y}, t)$ , *i.e.* the product of the results of simultaneous measurement of the fields at  $\mathbf{x}$  and at  $\mathbf{y}$ . If no particle is present, the average of many repeats of this measurement is

$$\langle 0 | \phi(\mathbf{x}, t)\phi(\mathbf{y}, t) | 0 \rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\hbar}{\omega(\mathbf{k})} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}.$$

Note that due to the  $1/\omega(\mathbf{k})$ , this is not 0 even when the fields are not measured at the same point. This implies a correlation between the measurements at separate points, but does not contradict causality:  $[\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = 0$ ! The vacuum is an energy eigenstate which requires an infinite time to set up. Thus there is plenty of time to set up correlations at distant points.

Now, suppose a particle is present in the wave packet  $f$ , and the same quantity is measured.

$$\langle f | \phi(\mathbf{x}, t)\phi(\mathbf{y}, t) | f \rangle = \langle 0 | \phi(\mathbf{x}, t)\phi(\mathbf{y}, t) | 0 \rangle + \psi^*(\mathbf{x}, t)\psi(\mathbf{y}, t) + \psi^*(\mathbf{y}, t)\psi(\mathbf{x}, t).$$

The presence of the particle causes the change in average results given by the last 2 terms on the r.h.s. which are modulated by the function

$$\begin{aligned} \psi(\mathbf{x}, t) &\equiv \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\omega(\mathbf{k})}} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} f(\mathbf{k}) \\ &\approx \psi(\mathbf{x} - \mathbf{v}_g t, 0) \\ &\approx \sqrt{\frac{\hbar}{2\omega(\mathbf{k}_0)}} \psi_S(\mathbf{x} - \mathbf{v}_g t, 0). \end{aligned}$$

The last two forms hold if the packet function  $f$  is narrowly peaked about  $\mathbf{k}_0$ .  $\psi_S$  is the one particle

Schrödinger wave function corresponding to momentum wave function  $f$ .  $\mathbf{v}_g \equiv \partial\omega/\partial\mathbf{k}|_{\mathbf{k}=\mathbf{k}_0}$  is the usual group velocity. The disturbance is nonzero only when the point is in the support of the wave function.

### 3.2. INTERACTING SCALAR FIELD THEORY

So far we have only considered linear field equations, whose associated Lagrangians are quadratic functions of the field. As soon as the field equations become non-linear, the particles associated with the quantum fields will interact with each other. In particular they can scatter.

The simplest way to introduce non-linear terms in the field equations, while preserving Lorentz invariance, is to add Lorentz scalar terms cubic and higher in the fields to the Lagrangian density. For scalar fields this is easy to do. For example any term

$$-\frac{g_n}{n!}\phi^n(x), \quad \text{for } n > 2$$

is a scalar which produces a term  $g_n\phi^{n-1}/(n-1)!$  in the field equation. Since such terms do not involve time derivatives, they do not alter the definition of the conjugate momentum nor the commutation relations. But clearly plane waves no longer solve the new field equations so there will be interactions.

Nonlinear quantum field theory is too complicated to solve exactly, so we shall first assume the nonlinear terms are small so they can be handled using perturbation theory. In the next chapter we review the formalism of time dependent perturbation theory.

## 4. Time Dependent Perturbation Theory

Although there are special quantum field theories, for example some in 2 space-time dimensions, that can be solved exactly, the exact solution of realistic interacting quantum field theories in 4 space-time dimensions is beyond reach. There are important cases for which linear QFT's in the presence of certain external fields can be solved. For example the Dirac equation in a Coulomb potential admits an exact solution for which one can find all the energy eigenvalues and eigenstates. The Coulomb potential is of special importance since the exact solution in that case is the starting point for the relativistic theory of atomic energy levels. Still the complete dynamics is never exactly given by these special cases and perturbation theory is the important tool for evaluating corrections to the exactly soluble (idealized) case, which can be free field theory with no external fields or one of the above cases. When we quantize the electromagnetic field, perturbation theory is essentially our *only* tool for computing radiative corrections due to the quantum nature of the electromagnetic field.

### 4.1. HEISENBERG AND SCHRÖDINGER PICTURES

In the Schrödinger picture the quantum dynamics is given by the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Phi, t\rangle = H_S(t) |\Phi, t\rangle$$

where we stress that we allow time varying external forces to be present (hence the time dependence of  $H$ ). The field equations arise in the Heisenberg picture wherein the time dependence resides in the operators rather than in the system states which are constant in time. To pass to the Heisenberg picture we write

$$|\Phi, t\rangle = U(t) |\Phi, 0\rangle$$

where

$$i\hbar \frac{\partial}{\partial t} U = H_S(t) U \quad U(0) = I,$$

and give the time independent Schrödinger picture operators  $\Omega$  time dependence according to

$$\Omega(t) \equiv U^\dagger \Omega U.$$

The Heisenberg picture Hamiltonian is similarly related to the Schrödinger picture one by

$$H(t) \equiv U^\dagger H_S(t) U.$$

Then the Heisenberg picture operators corresponding to constant Schrödinger picture operators satisfy the Heisenberg equations

$$i\hbar \dot{\Omega}(t) = [\Omega(t), H(t)].$$

It is most natural to formulate the time dependence in quantum field theory using Heisenberg picture, since the field operators will then satisfy equations of motion that are the direct quantum analogue of the classical field equations. We shall therefore always understand  $H(t)$  without subscripts to be the Hamiltonian in Heisenberg picture. When we work with any other picture we will attach a subscript to  $H$ , e.g.  $H_S(t)$  is the Hamiltonian in Schrödinger picture.

## 4.2. ASYMPTOTIC STATES AND MATRIX ELEMENTS

In discussing time dependent processes, it is convenient to introduce asymptotic states which are eigenstates of  $H(\pm\infty)$ . We denote by  $|in\rangle$  the ground state of  $H(-\infty)$  and by  $|out\rangle$  the ground state of  $H(+\infty)$ . The normal situation will be one in which all external fields vanish at sufficiently early and late times. Thus  $|in\rangle$  and  $|out\rangle$  will typically be ground states of  $H_0(-\infty)$  and  $H_0(+\infty)$  respectively. Although these operators are not the same (because their time evolution is governed by  $H$  not  $H_0$ ), the spectra of the two Hamiltonians *are* identical:  $H_0(t) = U^{-1}(t, -\infty)H_0S U(t, -\infty)$ . By convention we are identifying the Schrödinger and Heisenberg pictures at  $t = -\infty$ . Thus, if  $|in\rangle$  is the ground state of  $H_0(-\infty) = H_0S$ , the state  $\langle in|U(\infty, -\infty)$  is an eigenstate of  $H_0(+\infty)$  with the same eigenvalue and hence the ground state. Thus we can and shall fix phases by defining

$$\langle out| \equiv \langle in|U(\infty, -\infty).$$

We stress that this is the true “out” state only when  $H_S(\infty) = H_S(-\infty) \equiv H_0$ .

If the time dependence of  $H_S$  is *adiabatic*, *i.e.* very slow on the time scale set by the level spacings, the Adiabatic Theorem assures us that an eigenstate of  $H_S(-\infty)$  evolves to an eigenstate of  $H_S(t)$  for all  $t$  for which adiabatic conditions apply, even after a long enough time to change  $H_S$  by a finite amount. For example, the state  $|in\rangle$  will be an eigenstate of  $H(t)$  for all  $t$  for which adiabatic time variation applies. In particular, the ground state eigenvalue  $E_G(t)$  must not get close to the next higher eigenvalue as  $t$  varies. If this situation holds for all time, it follows that the state  $|in\rangle$  is a phase times the state  $|out\rangle$ , or  $\langle out|$  is this same phase times  $\langle in|$ . This phase is easily evaluated in terms of the time dependent ground state energy  $E_G(t)$  of  $H_S(t)$  by applying the Schrödinger equation to  $\langle in|U(t, -\infty)|in\rangle$  and using the adiabatic theorem  $H_S(t)U(t, -\infty)|in\rangle = E_G(t)U(t, -\infty)|in\rangle$ :

$$\langle out|in\rangle = \exp \left\{ -i \int_{-\infty}^{\infty} dt E_G(t) \right\} \quad \text{Adiabatic Conditions.}$$

Note carefully that adiabatic conditions would *not* apply if the ground state energy got close to an excited level as time evolved. In particular, it would not apply in processes with pair production when  $|\langle out|in\rangle| < 1$ .

## 4.3. GENERAL FORMALISM

We shall keep the initial discussion completely general and consider the situation in which the Heisenberg picture Hamiltonian is the sum of two pieces,

$$H(t) = H_0(t) + H'(t)$$

where  $H_0$  can be exactly dealt with and  $H'$  is “small” in an appropriate sense. Note that even when  $H$  is independent of time,  $H_0$  and  $H'$  still depend on time through the time dependence of the Heisenberg operators that enter it. The Heisenberg equations for the dynamical variables have the form

$$i \frac{\partial \Omega}{\partial t} = [\Omega, H(t)].$$

The goal of time dependent perturbation theory is to expand the evolution operator  $U(t, t_0)$  which carries the time dependence of the Heisenberg picture operators  $\Omega = U^{-1}\Omega_S U$ , or alternatively the time dependence of

the Schrödinger picture system states  $|\psi, t\rangle = U|\psi, t_0\rangle$ , in a power series in  $H'$ . A complication is that there is time dependence in  $U$  even when the perturbation vanishes. To systematically deal with this complication a new Interaction Picture (sometimes called the Dirac Picture) has been devised in which the operators carry the (known) time dependence due to  $H_0$  and the perturbation only enters the modified evolution operator  $U_I$ , which is constructed to be the identity in the absence of  $H'$ .

Thus for each Heisenberg picture operator with no explicit time dependence, we define an interaction picture operator by

$$\Omega_I(t) \equiv U_I(t)\Omega(t)U_I^{-1}(t) \quad (4.1)$$

and require that  $\Omega_I$  satisfies the Heisenberg equation with Hamiltonian  $H_{0I} = U_I H_0(t) U_I^{-1}$ \*

$$i\frac{\partial\Omega_I}{\partial t} = [\Omega_I, H_{0I}(t)].$$

differentiating (4.1) we find the requirement

$$\begin{aligned} [\Omega_I, H_{0I}(t)] &= i\dot{U}_I\Omega U_I^{-1} + iU_I\Omega\dot{U}_I^{-1} + U_I[\Omega, H(t)]U_I^{-1} \\ &= i\dot{U}_I U_I^{-1}\Omega_I + i\Omega_I U_I\dot{U}_I^{-1} + [\Omega_I, H_{0I}(t)] + [\Omega_I, H'_I(t)] \\ &= [\Omega_I, H_{0I}(t)] + [\Omega_I, H'_I(t) - i\dot{U}_I U_I^{-1}]. \end{aligned}$$

Thus the equation for  $U_I$  is just

$$i\dot{U}_I(t) = H'_I(t)U_I(t) = U_I(t)H'(t).$$

We choose the initial condition  $U_I(t_0) = I$ , in which case it is a good idea to display two time arguments  $U_I(t, t_0)$  as we did for  $U$ . Notice that since  $U$  relates Heisenberg and Schrödinger pictures, the equation ( ) for  $U$  can be also written

$$i\dot{U} = UH(t) \quad (4.2)$$

from which it is clear that we can express  $U = U_0 U_I$  where

$$i\dot{U}_0 = U_0 H_{0I}(t)$$

To expand  $U_I$  in powers of  $H'$  it is convenient first to incorporate initial condition information by writing the integral equation

$$U_I(t, t_0) = I - i \int_{t_0}^t dt' H'_I(t') U_I(t', t_0), \quad (4.3)$$

and then to generate the perturbation series by iteration

$$\begin{aligned} U_I(t, t_0) &= I - i \int_{t_0}^t dt' H'_I(t') (I - i \int_{t_0}^{t'} dt'' H'_I(t'') U_I(t'', t_0)) \\ &= I - i \int_{t_0}^t dt' H'_I(t') + (-i)^2 \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' H'_I(t') H'_I(t'') + \dots \end{aligned}$$

There is a useful way to summarize the entire perturbation series, which employs the concept of the time ordered product of operators. Consider a set of operators each associated with a different time,

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\* Note that if  $H_0 = H_0(\Omega_k(t), t)$ , then  $H_{0I} = H_0(\Omega_{Ik}(t), t)$ .

$A_1(t_1), A_2(t_2), \dots, A_N(t_N)$ . The time ordered product of these operators is defined as the ordinary product with the operators ordered according to the time argument: the operator  $A_k(t_k)$  to the left of  $A_l(t_l)$  if  $t_k > t_l$ . If there are any anticommuting operators in the set, there is also an overall  $-1$  if one achieves the time ordering by an odd permutation of fermionic operators. Thus, for example,

$$T[A(t_1)B(t_2)] = \begin{cases} A(t_1)B(t_2) & t_1 > t_2 \\ \pm B(t_2)A(t_1) & t_2 > t_1 \end{cases}$$

with the  $-$  for both operators fermionic. Now the factors of  $H'_I$  in the series for  $U_I$  are all time ordered due to the limits of integration. If we make use of the time ordering symbol, we can extend all integrations to the full range  $t_0 < t' < t$  provided we divide the  $n^{\text{th}}$  term by  $n!$  to account for the overcounting due to the  $n!$  orderings of the  $t$ 's. Thus the entire series becomes

$$U_I(t, t_0) = \sum_{n=0}^{\infty} \frac{1}{n!} (-i)^n \int_{t_0}^t dt_1 dt_2 \cdots dt_n T[H'_I(t_1)H'_I(t_2) \cdots H'_I(t_n)].$$

If it weren't for the time ordering symbol this would be just the exponential series. It is therefore a useful mnemonic to write

$$U_I(t, t_0) = T e^{-i \int_{t_0}^t dt' H'_I(t')} \quad (4.4)$$

where it is understood that  $t > t_0$ . This equation is known as the Dyson Formula. Since the formula just reflects the equation  $U_I$  satisfies, we can write a similar formula for the full  $U$ :

$$U(t, t_0) = T e^{-i \int_{t_0}^t dt' H_S(t')} = \bar{T} e^{-i \int_{t_0}^t dt' H(t')} = \bar{T} e^{-i \int_{t_0}^t dt' H_{0,I}(t')} T e^{-i \int_{t_0}^t dt' H'_I(t')}$$

where  $\bar{T}$  denotes anti-time ordering (later times to the *right*).

It will be useful to extend the definition of  $U_I(t, t_0)$  to times earlier than  $t_0$ . We shall do this in a way to preserve the *closure* property

$$U_I(t, t_1)U_I(t_1, t_0) = U_I(t, t_0), \quad (4.5)$$

which follows from the differential equation and initial condition for  $t > t_1 > t_0$ . If we set  $t = t_0$  in (4.5), the r.h.s. is just  $I$  so we have to define

$$U_I(t, t_0) \equiv U_I^{-1}(t_0, t) = U_I^\dagger(t_0, t) \quad \text{for } t < t_0.$$

(Note that  $U_I^\dagger U_I = I$  is a simple consequence of the differential equation and the hermiticity of  $H'_I$ .) It is then simple to check that (4.5) holds for all time orderings.

Next let us show how to express various physical quantities in the Interaction picture. One interesting quantity is the so called vacuum persistence amplitude  $\langle out|in \rangle$ , given by

$$\langle out|in \rangle = \langle in|U(\infty, -\infty)|in \rangle = \langle in|U_0(\infty, -\infty)U_I(\infty, -\infty)|in \rangle.$$

We shall identify all pictures at  $t = -\infty$ . In the usual situation where external fields vanish at early times the state  $|in \rangle$  will be the ground state of  $H_0(-\infty) = H_{0I}$ . Furthermore, with no external fields in  $H_{0I}(t)$

the latter will be time independent for all time (since its time evolution is governed by  $H_{0I}$  itself). In this situation

$$U_0(t, t_0) = e^{-iH_{0I}(t-t_0)}$$

and  $\langle in|$  is an eigenstate of  $U_0(t, t_0)$  with eigenvalue  $e^{-iE_0(t-t_0)}$ , where  $E_0$  is the ground state energy of  $H_{0I}$ . By **convention** we can choose our zero of energy so that  $E_0 = 0$ , in which case we have

$$\langle out|in\rangle = \langle 0, I|U_I(\infty, -\infty)|0, I\rangle \quad E_0 = 0.$$

The persistence amplitude carries a lot of information, because it can be defined for any choice of external fields. Its dependence on these external fields can then be exploited to obtain numerous matrix elements relevant to the zero field situation. We shall see many applications of this remark in the course of our studies. One can also get the energy spectrum for static external fields from this amplitude by switching them on at some early time keeping them constant for a long time  $2T$  and then switching them off. The  $T$  dependence of  $\langle out|in\rangle$  will then display the dependence  $e^{-iE_k 2T}$  from which the energy eigenvalues can be read off. The states that are probed by this device will depend on the manner of the switching on procedure. For adiabatic switching on, only the ground state in the presence of the static field will contribute.

#### 4.4. SCATTERING IN AN EXTERNAL FIELD: BORN APPROXIMATION

One can equally well choose initial and final states that contain particles. For example, the amplitude for a scalar particle with momentum  $\mathbf{p}$  initially making a transition to  $\mathbf{p}'$  at very late times is

$$\begin{aligned} \langle 0, I|a(\mathbf{p}')U_I(\infty, -\infty)a^\dagger(\mathbf{p})|0, I\rangle &\approx \\ &\delta(\mathbf{p}' - \mathbf{p}) - i \int_{-\infty}^{\infty} dt \langle 0, I|a(\mathbf{p}')H'_I(t)a^\dagger(\mathbf{p})|0, I\rangle, \end{aligned}$$

where we kept only terms to first order. As a concrete example, consider the interaction  $\int dt H'_I(t) = -\int d^4x \phi_I^2(x)B(x)/2$ , where  $B(x)$  is a fixed external scalar field. Since the  $\phi_I$  are free fields, they can be expressed in terms of creation and annihilation operators and the matrix element evaluated:

$$\begin{aligned} &\int_{-\infty}^{\infty} dt \langle 0, I|a(\mathbf{p}')H'_I(t)a^\dagger(\mathbf{p})|0, I\rangle \\ &= - \int d^4x B(x) e^{i(p-p')\cdot x} \frac{1}{(2\pi)^3 \sqrt{2\omega(\mathbf{p})} \sqrt{2\omega(\mathbf{p}')}} \\ &= - \frac{1}{(2\pi)^3 \sqrt{2\omega(\mathbf{p})} \sqrt{2\omega(\mathbf{p}')}} \tilde{B}(p' - p), \end{aligned}$$

where  $\tilde{B}(q) = \int d^4x B(x)e^{-iq\cdot x}$  is the Fourier transform of the external field. Note that in the case  $B$  is static the time integral gives a factor of  $2\pi\delta(\omega' - \omega)$ .

Recall from basic scattering theory that if the scattering matrix for a particle from a static potential is

written

$$\langle \mathbf{q}, out | \mathbf{p}, in \rangle = \delta(\mathbf{q} - \mathbf{p}) - 2\pi i \delta(\omega(\mathbf{q}) - \omega(\mathbf{p})) T(\mathbf{q}, \mathbf{p})$$

then the differential scattering cross section is given by

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{d^3q}{d\Omega} \delta(\omega(\mathbf{q}) - \omega(\mathbf{p})) \frac{(2\pi)^4}{v} |T(\mathbf{q}, \mathbf{p})|^2, \\ &= q^2 dq \delta(\omega(\mathbf{q}) - \omega(\mathbf{p})) \frac{(2\pi)^4}{v} |T(\mathbf{q}, \mathbf{p})|^2, \\ &= p\omega(\mathbf{p}) \frac{(2\pi)^4}{v} |T(\mathbf{q}, \mathbf{p})|^2, \\ &= \omega(\mathbf{p})^2 (2\pi)^4 |T(\mathbf{q}, \mathbf{p})|^2, \end{aligned}$$

where  $v$  is the speed of the incident particle. Defining the *spatial* Fourier transform  $\tilde{B}(\mathbf{k}) \equiv \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} B(\mathbf{x})$ , we then obtain

$$T^{Born}(\mathbf{q}, \mathbf{p}) = -\frac{1}{(2\pi)^3 2\omega(\mathbf{p})} \tilde{B}(\mathbf{q} - \mathbf{p}),$$

giving the cross section

$$\frac{d\sigma^{Born}}{d\Omega} = \frac{1}{16\pi^2} |\tilde{B}(\mathbf{q} - \mathbf{p})|^2.$$

Another scattering example, the amplitude for a Dirac particle with momentum and helicity  $\mathbf{p}, \lambda$  initially making a transition to  $\mathbf{p}', \lambda'$  at very late times is

$$\begin{aligned} \langle 0, I | b_{\lambda'}(\mathbf{p}') U_I(\infty, -\infty) b_{\lambda}^{\dagger}(\mathbf{p}) | 0, I \rangle &\approx \\ &\delta_{\lambda'\lambda} \delta(\mathbf{p}' - \mathbf{p}) - i \int_{-\infty}^{\infty} dt \langle 0, I | b_{\lambda'}(\mathbf{p}') H'_I(t) b_{\lambda}^{\dagger}(\mathbf{p}) | 0, I \rangle, \end{aligned}$$

where we kept only terms to first order. In the case of a weak external electromagnetic field,  $\int dt H'_I(t) = -\int d^4x j_I^{\mu}(x) A_{\mu}(x)$ , where  $j_I^{\mu} = \frac{q}{2} [\bar{\psi}_I, \gamma^{\mu} \psi_I]$ . Since the  $\psi_I$  are free fields, they can be expressed in terms of creation and annihilation operators and the matrix element evaluated:

$$\begin{aligned} &\int_{-\infty}^{\infty} dt \langle 0, I | b_{\lambda'}(\mathbf{p}') H'_I(t) b_{\lambda}^{\dagger}(\mathbf{p}) | 0, I \rangle \\ &= -q \int d^4x A_{\mu}(x) e^{i(p-p')\cdot x} \frac{1}{(2\pi)^3 \sqrt{2\omega(\mathbf{p})} \sqrt{2\omega(\mathbf{p}')}} \bar{u}_{\lambda'}(\mathbf{p}') \gamma^{\mu} u_{\lambda}(\mathbf{p}) \\ &= -q \frac{1}{(2\pi)^3 \sqrt{2\omega(\mathbf{p})} \sqrt{2\omega(\mathbf{p}')}} \bar{u}_{\lambda'}(\mathbf{p}') \gamma^{\mu} \tilde{A}_{\mu}(p' - p) u_{\lambda}(\mathbf{p}), \end{aligned}$$

where  $\tilde{A}(q) = \int d^4x A(x) e^{-iq\cdot x}$  is the Fourier transform of the potential. Note that in the case  $A$  is static the time integral gives a factor of  $2\pi\delta(\omega' - \omega)$ .

Recall from basic scattering theory that if the scattering matrix for a particle from a static potential is written

$$\langle \mathbf{q}, out | \mathbf{p}, in \rangle = \delta_{\lambda'\lambda} \delta(\mathbf{q} - \mathbf{p}) - 2\pi i \delta(\omega(\mathbf{q}) - \omega(\mathbf{p})) T_{\lambda'\lambda}(\mathbf{q}, \mathbf{p})$$

then the differential scattering cross section is given by

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{d^3q}{d\Omega} \delta(\omega(\mathbf{q}) - \omega(\mathbf{p})) \frac{(2\pi)^4}{v} |T_{\lambda'\lambda}(\mathbf{q}, \mathbf{p})|^2, \\
&= q^2 dq \delta(\omega(\mathbf{q}) - \omega(\mathbf{p})) \frac{(2\pi)^4}{v} |T_{\lambda'\lambda}(\mathbf{q}, \mathbf{p})|^2, \\
&= p\omega(\mathbf{p}) \frac{(2\pi)^4}{v} |T_{\lambda'\lambda}(\mathbf{q}, \mathbf{p})|^2, \\
&= \omega(\mathbf{p})^2 (2\pi)^4 |T_{\lambda'\lambda}(\mathbf{q}, \mathbf{p})|^2,
\end{aligned}$$

where  $v$  is the speed of the incident particle.

Defining  $\tilde{A}(\mathbf{k}) \equiv \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} A(\mathbf{x})$ , we then obtain

$$T_{\lambda'\lambda}^{Born}(\mathbf{q}, \mathbf{p}) = -\frac{1}{(2\pi)^3 2\omega(\mathbf{p})} \bar{u}_{\lambda'} q \gamma \cdot \tilde{A}(\mathbf{q} - \mathbf{p}) u_{\lambda}(\mathbf{p}),$$

giving the cross section

$$\frac{d\sigma^{Born}}{d\Omega} = \frac{1}{16\pi^2} |\bar{u}_{\lambda'} q \gamma \cdot \tilde{A}(\mathbf{q} - \mathbf{p}) u_{\lambda}(\mathbf{p})|^2.$$

Consider the example of the scattering of an electron with  $q = -e$  from the Coulomb potential of a nucleus of atomic number  $Z$  (Mott Scattering),  $A^0 = Ze/4\pi r$ ,  $\mathbf{A} = 0$ . Then  $\tilde{A}^0(\mathbf{k}) = Ze/k^2$  and

$$\frac{d\sigma^{Born}}{d\Omega} = \frac{e^4 Z^2}{16\pi^2 (\mathbf{q} - \mathbf{p})^4} |\bar{u}_{\lambda'}(\mathbf{q}) \gamma^0 u_{\lambda}(\mathbf{p})|^2 = \frac{\alpha^2 Z^2}{(\mathbf{q} - \mathbf{p})^4} |\bar{u}_{\lambda'}(\mathbf{q}) \gamma^0 u_{\lambda}(\mathbf{p})|^2.$$

Here we have introduced the fine structure constant  $\alpha = e^2/4\pi \approx 1/137$ . Evaluating the spinor matrix element in terms of two component helicity spinors leads to (using  $|\mathbf{q}| = |\mathbf{p}|$ )

$$\bar{u}_{\lambda'}(\mathbf{q}) \gamma^0 u_{\lambda}(\mathbf{p}) = \left[ \omega(\mathbf{p}) + m + \frac{4\lambda\lambda' \mathbf{p}^2}{\omega + m} \right] \chi_{\lambda'}^{\dagger}(\mathbf{q}) \chi_{\lambda}(\mathbf{p}).$$

The absolute square of  $\chi^{\dagger} \chi$  can be evaluating by noting that the  $2 \times 2$  matrix  $\chi \chi^{\dagger}$  is a projector onto the spin state of definite helicity:

$$\chi_{\lambda}(\mathbf{p}) \chi_{\lambda}^{\dagger}(\mathbf{p}) = \frac{1 + 2\lambda \hat{\mathbf{p}} \cdot \boldsymbol{\sigma}}{2}.$$

Thus we have

$$\begin{aligned}
|\chi_{\lambda'}^{\dagger}(\mathbf{q}) \chi_{\lambda}(\mathbf{p})|^2 &= \frac{1}{4} \text{tr}[(1 + 2\lambda \hat{\mathbf{p}} \cdot \boldsymbol{\sigma})(1 + 2\lambda' \hat{\mathbf{q}} \cdot \boldsymbol{\sigma})] \\
&= \frac{1}{2} (1 + 4\lambda\lambda' \hat{\mathbf{p}} \cdot \hat{\mathbf{q}})
\end{aligned}$$

Inserting all this into the formula for the differential cross section, we obtain after simplifying

$$\frac{d\sigma^{Born}}{d\Omega} = \frac{\alpha^2 Z^2}{(\mathbf{q} - \mathbf{p})^4} [\omega^2 + m^2 + \mathbf{q} \cdot \mathbf{p} + 4\lambda\lambda' (\mathbf{p}^2 + (\omega^2 + m^2) \hat{\mathbf{p}} \cdot \hat{\mathbf{q}})]$$

To compare all of the details of this formula with experiment we would have to prepare a polarized beam of electrons with definite helicity and also measure the spin of the final electron. A noteworthy feature of such a complete experiment is that at high energies there is an overall factor of  $(1 + 4\lambda\lambda') = 2\delta_{\lambda\lambda'}$ , which means

that helicity is conserved at high energy. If we don't measure the final spin we should sum over  $\lambda' = \pm 1/2$  to obtain

$$\frac{d\sigma^{Born}}{d\Omega_{Unobserved\ spin}} = 2 \frac{\alpha^2 Z^2}{(\mathbf{q} - \mathbf{p})^4} [\omega^2 + m^2 + \mathbf{q} \cdot \mathbf{p}].$$

Similarly, if we have a completely unpolarized beam, we need to average over  $\lambda$  to obtain

$$\frac{d\sigma^{Born}}{d\Omega_{unpol}} = \frac{\alpha^2 Z^2}{(\mathbf{q} - \mathbf{p})^4} [\omega^2 + m^2 + \mathbf{q} \cdot \mathbf{p}],$$

independent of the final spin.

Two simplifying limits can be considered. The *nonrelativistic* or *low energy* limit  $\mathbf{p}^2 \ll m^2$  is

$$\frac{d\sigma^{NR}}{d\Omega} \sim 2m^2 \frac{\alpha^2 Z^2}{(\mathbf{q} - \mathbf{p})^4} [1 + 4\lambda\lambda' \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}]$$

Apart from the helicity dependence due to the spin of the electrons this is just the Rutherford formula. The opposite limit, the *ultrarelativistic* or *high energy* limit  $\mathbf{p}^2 \gg m^2$  is (assume  $\hat{\mathbf{p}} \cdot \hat{\mathbf{q}} \neq -1$ )

$$\frac{d\sigma^{UR}}{d\Omega} \sim \frac{\alpha^2 Z^2}{(\mathbf{q} - \mathbf{p})^4} [\mathbf{p}^2 + \mathbf{q} \cdot \mathbf{p}] [1 + 4\lambda\lambda'].$$

where the high energy helicity conservation is transparent.

#### 4.5. PAIR PRODUCTION IN A TIME VARYING EXTERNAL FIELD

Let us return to the scalar field external field perturbation  $H_I'(t) = -\int d^3x \phi_I^2 B(x)/2$  in the case where the initial state is the ground state of  $H_{0,I}$ ,  $|0\rangle_I$ , and the final state contains two particles:  $\langle 0, I | a(\vec{p}_1) a(\vec{p}_2)$ . The transition amplitude to first order is

$$T_{fi} = \frac{i}{2} \int d^4x \langle 0, I | a(\vec{k}_1) a(\vec{k}_2) \phi_I^2(x) B(x) / 2 | 0, I \rangle = \frac{i}{(2\pi)^3 2\sqrt{\omega_1 \omega_2}} \int d^4x e^{-i(k_1 + k_2) \cdot x} B(x) \equiv \frac{i\tilde{B}(k_1 + k_2)}{(2\pi)^3 2\sqrt{\omega_1 \omega_2}}$$

Squaring and integrating over final momenta we get the total pair production probability

$$P_{\text{pair}} = \frac{1}{2} \int \frac{d^3k_1 d^3k_2}{(2\pi)^6 4\omega_1 \omega_2} |\tilde{B}(k_1 + k_2)|^2 = \frac{1}{4(2\pi)^5} \int d^4K \theta(-K^2 - 4m^2) \sqrt{1 + \frac{4m^2}{K^2}} |\tilde{B}(K)|^2$$

since  $\omega_1 + \omega_2 > 2m$ ,  $P_{\text{pair}} \neq 0$  only when  $B$  oscillates with frequencies greater than  $2m$ . In particular a static external field will not produce pairs in perturbation theory.

#### 4.6. PERTURBATION THEORY FOR TIME ORDERED PRODUCTS

Another class of quantities that will be very useful to us is the matrix element of the time ordered product of a finite number of Heisenberg picture fields between asymptotic states:

$$\langle out | T[A_1(t_1) A_2(t_2) \cdots A_N(t_N)] | in \rangle.$$

The simplest way to transcribe this matrix element to interaction picture is to first assume the ordering

$t_1 > t_2 \cdots > t_N$  so that the  $T$  symbol can be removed. Then

$$\begin{aligned}
& \langle out | A_1(t_1) A_2(t_2) \cdots A_N(t_N) | in \rangle \\
&= \langle 0, I | U_I(\infty, -\infty) U_I^{-1}(t_1, -\infty) A_{I1}(t_1) U_I(t_1, -\infty) U_I^{-1}(t_2, -\infty) A_{I2}(t_2) \\
&\quad U_I(t_2, -\infty) \cdots U_I^{-1}(t_N, -\infty) A_{IN}(t_N) U(t_N, -\infty) | 0, I \rangle \\
&= \langle 0, I | U_I(\infty, t_1) A_{I1}(t_1) U_I(t_1, t_2) A_{I2}(t_2) U_I(t_2, t_3) \cdots U_I(t_{N-1}, t_N) A_{IN}(t_N) \\
&\quad U_I(t_N, -\infty) | 0, I \rangle,
\end{aligned}$$

where use has been made of the closure property of  $U_I$ . Now we notice that all of the interaction picture operators that appear in the final matrix element, including those in the Dyson formula for each  $U_I$  are time ordered. Thus if we insert the time ordering symbol in front of all the operators we can combine all of the  $U_I$ 's into a single  $U_I(\infty, -\infty)$  arriving at

$$\begin{aligned}
& \langle out | T[A_1(t_1) A_2(t_2) \cdots A_N(t_N)] | in \rangle \\
&= \langle 0, I | T[U_I(\infty, -\infty) A_{I1}(t_1) A_{I2}(t_2) \cdots A_{IN}(t_N)] | 0, I \rangle.
\end{aligned}$$

Finally, we simply note that had the time ordering been any other, the same steps would have led to the same final result.

In the usual situation where  $H_{0I}$  is the Hamiltonian for free fields, all of the interaction picture operators are free fields, and to evaluate each finite order in perturbation theory one only needs to master the computation of the vacuum expectation values of the time ordered product of a finite number of free fields. Free fields can always be expressed as a linear functional of creation and annihilation operators. Thus if  $\phi_k(x)$  is a free field, it can be written

$$\phi_k(x) = \phi_k^+(x) + \phi_k^-(x)$$

where  $\phi_k^+(x)$  annihilates  $|0\rangle$  and  $\phi_k^-(x)$  annihilates  $\langle 0|$ . Thus we have

$$\begin{aligned}
& \langle 0 | T[\phi_k(x) \phi_l(x')] | 0 \rangle \\
&= \theta(t - t') \langle 0 | \phi_k^+(x) \phi_l^-(x') | 0 \rangle \pm \theta(t' - t) \langle 0 | \phi_l^+(x') \phi_k^-(x) | 0 \rangle \\
&\equiv \theta(t - t') C_{kl}(x - x') \pm \theta(t' - t) C_{lk}(x' - x)
\end{aligned}$$

where  $C_{kl}(x - x') = [\phi_k^+(x), \phi_l^-(x')]_{\pm}$  is a  $c$  number since the fields are free.

Now consider a general time ordered product of  $N$  free fields

$$\langle 0 | T[\phi_1(x_1) \cdots \phi_N(x_N)] | 0 \rangle$$

and first assume  $t_1 > t_2 > \cdots > t_N$ . Then the leftmost field is  $\phi_1$  and it can be replaced by its annihilation part  $\phi_1^+$ , which is then moved via the commutation relations all the way to the right where it kills the vacuum. The (anti)commutators

$$[\phi_1^+(x_1), \phi_k(x_k)]_{\pm} = [\phi_1^+(x_1), \phi_k^-(x_k)]_{\pm} = \langle 0 | T[\phi_1(x_1) \phi_k(x_k)] | 0 \rangle$$

since the (anti)commutators are  $c$  numbers and  $t_1 > t_k$  by assumption. Thus

$$\begin{aligned}
& \langle 0 | T[\phi_1(x_1) \cdots \phi_N(x_N)] | 0 \rangle = \langle 0 | T[\phi_1(x_1) \phi_2(x_2)] | 0 \rangle \langle 0 | T[\phi_3(x_3) \cdots \phi_N(x_N)] | 0 \rangle \\
&\quad \pm \langle 0 | T[\phi_1(x_1) \phi_3(x_3)] | 0 \rangle \langle 0 | T[\phi_2(x_2) \phi_4(x_4) \cdots \phi_N(x_N)] | 0 \rangle \pm \cdots
\end{aligned} \tag{4.6}$$

where the sign in front of each term is dictated by the number of times the order of fermionic operators is switched. The time ordering symbol is not needed with our assumed ordering of times. But now we notice

that if the time ordering had been any other the same steps would have led to the same result *provided* we keep the  $T$  symbol in place. Thus we have related the vacuum expectation value of the time ordered product of  $N$  free fields to those of 2 free fields and  $N - 2$  free fields. By induction we can therefore express the vacuum expectation value of the time ordered product of  $N$  free fields as sums of products of the vacuum expectation values of the time ordered product of pairs of free fields.

The result, known as Wick's Theorem, can be expressed as follows. First note that the answer is 0 unless  $N$  is even. Then the vacuum expectation value of the time ordered product of  $N$  free fields is the sum of terms, one for each distinct pairing off of all the  $N$  fields. The term for each such pairing off is simply  $\pm$  the product of the vacuum expectation values of the time ordered product of each pair of fields in the given pairing off. The sign is determined by comparing the ordering of the  $N$  operators in the original time ordered product with the order they appear in the given term after being paired off. If the latter ordering is achieved by an odd permutation of fermionic operators the sign is  $-$ ; otherwise it is  $+$ . It doesn't matter what order we display the factors within a given term, since switching their order would always be an even permutation: a pairing of a boson field with a fermion field would always contribute zero!

#### 4.7. A TECHNICAL COMMENT ON TIME DERIVATIVES IN TIME DEPENDENT PERTURBATION THEORY

It is important to appreciate some subtle differences between time derivatives of operators in different pictures. For example, interaction picture depends on a specific breakup of the Heisenberg picture Hamiltonian  $H = H_0 + H'$ , so in Heisenberg picture

$$\dot{\Omega} = \frac{1}{i}[\Omega, H_0] + \frac{1}{i}[\Omega, H'].$$

The transformation to interaction picture, being a purely algebraic similarity transformation shows that

$$\begin{aligned} (\dot{\Omega})_I &= \frac{1}{i}[\Omega_I, H_{0I}] + \frac{1}{i}[\Omega_I, H'_I] \\ &= \dot{\Omega}_I + \frac{1}{i}[\Omega_I, H'_I], \end{aligned}$$

so there is in general a discrepancy between the interaction picture operator corresponding to  $\dot{\Omega}$  and the time derivative of the operator  $\Omega_I$ . When we use the Dyson formula for time dependent perturbation theory to calculate a matrix element involving  $\dot{\Omega}$ , care must be taken about this difference. However there is a very simple prescription to keep things straight. This is to always think of time derivatives of operators in the Dyson formula as acting *outside* the time ordering symbol. Note the following identity

$$\begin{aligned} \frac{\partial}{\partial t} T[e^{-i \int_{-\infty}^{\infty} dt' H'_I(t')} \Omega_I(t)] &= T[e^{-i \int_t^{\infty} dt' H'_I(t')} (\dot{\Omega}_I(t) + \frac{1}{i}[\Omega_I(t), H'_I(t)]) T[e^{-i \int_{-\infty}^t dt' H'_I(t')}] \\ &= T[e^{-i \int_{-\infty}^{\infty} dt' H'_I(t')} (\dot{\Omega}(t))_I]. \end{aligned}$$

This comment becomes particularly useful in cases such as scalar electrodynamics where the relation between  $\pi^\dagger(x)$  and  $\dot{\phi}(x) = \pi^\dagger(x) + iQA_0\phi(x)$  involves the interaction. Since the interaction picture fields are free, the relationship in that picture is  $\dot{\phi}_I(x) = \pi_I^\dagger(x)$ . Thus the exponent in the Dyson formula shows a disquieting asymmetry between space and time:

$$\int dt H'_I(t) = \int d^4x \left( iQ\mathbf{A} \cdot (\phi_I^\dagger \nabla \phi_I - (\nabla \phi_I^\dagger) \phi_I) + Q^2 \mathbf{A}^2 \phi_I^\dagger \phi_I - iQA_0(\phi_I^\dagger \dot{\phi}_I - \dot{\phi}_I^\dagger \phi_I) \right).$$

However, it is possible to prove a "reshuffling theorem" that if all time derivatives in the Dyson formula are understood to be taken outside the time ordering symbol, covariance is restored. In other words there

are two sources of apparent non-covariance: the form of  $H'_I$  and the time ordering operation itself. To present the results of the reshuffling theorem, we introduce the symbol  $T^*$  to signify time ordering in which all time derivatives are taken outside the time ordering symbol. Then the reshuffling theorem for scalar electrodynamics can be stated

$$T[e^{-i \int_{-\infty}^{\infty} dt' H'_I(t')}] = T^* \left[ e^{-i \int d^4x (iQA^\mu (\phi_I^\dagger \partial_\mu \phi_I - (\partial_\mu \phi_I^\dagger) \phi_I) + Q^2 A^\mu A_\mu \phi_I^\dagger \phi_I)} \right].$$

When employing the Wick expansion to the r.h.s. one simply needs to remember that one never uses a quantity such as  $\langle 0|T[\partial_\mu \phi(x)\phi^\dagger(y)]|0\rangle$ , namely all derivatives occur outside not inside the time ordering symbols.

As an illuminating example of these ideas we quote the improved Dyson formula for the out-in matrix element of the current operator:

$$\langle out|j_\mu(x)|in\rangle = \langle 0,I|T^* \left[ e^{-i \int d^4x (iQA^\mu (\phi_I^\dagger \partial_\mu \phi_I - (\partial_\mu \phi_I^\dagger) \phi_I) + Q^2 A^\mu A_\mu \phi_I^\dagger \phi_I)} \right. \\ \left. (-iQ(\phi_I^\dagger \partial_\mu \phi_I - (\partial_\mu \phi_I^\dagger) \phi_I) - 2Q^2 A_\mu \phi_I^\dagger \phi_I) \right] |0,I\rangle.$$

Take particular note of the manifest covariance of the r.h.s. of this formula.

#### 4.8. PROPAGATORS FOR SCALAR AND DIRAC FIELDS

Wick's Theorem assures us that to obtain a general time ordered product of free fields, we only need to know the two field case,  $\langle 0|T[\phi_1(x_1)\phi_2(x_2)]|0\rangle$ , which is also called the two point function and sometimes the propagator.

Let us first work out the propagator for a free scalar field which has the representation

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{p})}} (a(\mathbf{p})e^{ip \cdot x} + b^\dagger(\mathbf{p})e^{-ip \cdot x}), \quad (4.7)$$

where  $a$  annihilates a particle and  $b^\dagger$  creates an antiparticle. These operators satisfy the commutation relations

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = [b(\mathbf{p}), b^\dagger(\mathbf{p}')] = \delta(\mathbf{p}' - \mathbf{p}),$$

with all other commutators vanishing<sup>\*</sup>. The Hamiltonian for the free scalar field is easy to write down

$$H = \int d^3p \omega(\mathbf{p})(a^\dagger(\mathbf{p})a(\mathbf{p}) + b^\dagger(\mathbf{p})b(\mathbf{p})) \\ = \int d^3x : (\dot{\phi}^\dagger \dot{\phi} + \nabla \phi^\dagger \cdot \nabla \phi + m^2 \phi^\dagger \phi) :$$

where the double colons  $:(\dots):$  denotes normal ordering, *i.e.* all creation operators to the left of all annihilation operators.

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\* Notice that the commutator

$$[\phi(x_1), \phi^\dagger(x_2)] = \int \frac{d^3p}{(2\pi)^3 2\omega(\mathbf{p})} (e^{ip \cdot (x_1 - x_2)} - e^{ip \cdot (x_2 - x_1)})$$

vanishes for space-like separations  $(x_2 - x_1)^2 > 0$ . To see this go to a Lorentz frame where  $t_2 = t_1$  (always possible for space-like separations). Then the second term cancels the first after the variable change  $\mathbf{p} \rightarrow -\mathbf{p}$ . If  $a, b$  satisfied anticommutation relations, the anticommutator would not have this locality property since the two terms would then add. This is the spin-statistics connection for scalar fields. Also notice that if the  $\omega(\mathbf{p})$  were absent anticommutation relations *would* be local.

Clearly the vacuum expectation of the time ordered product of two  $\phi$ 's or two  $\phi^\dagger$ 's vanishes, and

$$\begin{aligned} \langle 0|T[\phi(x_1)\phi^\dagger(x_2)]|0\rangle &= \theta(t_1 - t_2) \int \frac{d^3p}{(2\pi)^3 2\omega(\mathbf{p})} e^{ip \cdot (x_1 - x_2)} \\ &+ \theta(t_2 - t_1) \int \frac{d^3p}{(2\pi)^3 2\omega(\mathbf{p})} e^{ip \cdot (x_2 - x_1)}, \end{aligned} \quad (4.8)$$

where we recall that  $p \cdot x = \mathbf{p} \cdot \mathbf{x} - \omega t$ . To make this expression less unwieldy, it is helpful to use the following integral representation for the step function

$$\theta(t) = \int_{-\infty}^{\infty} \frac{dp^0}{2\pi i} e^{-ip^0 t} \frac{1}{-p^0 - i\epsilon} \quad \epsilon \rightarrow 0_+.$$

Including the factor  $e^{-i\omega t}$  gives

$$\theta(t)e^{-i\omega t} = \int_{-\infty}^{\infty} \frac{dp^0}{2\pi i} e^{-ip^0 t} \frac{1}{\omega - p^0 - i\epsilon},$$

after a shift of  $p^0$ . Inserting this representation into (4.8), gives

$$\langle 0|T[\phi(x_1)\phi^\dagger(x_2)]|0\rangle = \int \frac{d^4p}{(2\pi)^4 2\omega(\mathbf{p})} e^{ip \cdot (x_1 - x_2)} \frac{-i}{\omega - p^0 - i\epsilon} + e^{ip \cdot (x_2 - x_1)} \frac{-i}{\omega - p^0 - i\epsilon},$$

where now  $p \cdot x = \mathbf{p} \cdot \mathbf{x} - p^0 t$ . Thus we can change  $p \rightarrow -p$  in the second term and then combine it with the first to obtain finally

$$\langle 0|T[\phi(x_1)\phi^\dagger(x_2)]|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot (x_1 - x_2)} \frac{-i}{p^2 + m^2 - i\epsilon} \equiv \Delta_F(x_1 - x_2). \quad (4.9)$$

From its definition the propagator should have the property that only positive frequency components should be present as  $t_k \rightarrow +\infty$  and negative frequency components as  $t_k \rightarrow -\infty$ . This property is assured in (4.9) by the  $-i\epsilon$  in the denominator. The propagator is a Green function for the Klein-Gordon differential  $-\partial^2 + m^2$ :

$$(-\partial^2 + m^2)\Delta_F(x, y) = -i\delta(x - y).$$

The  $i\epsilon$  prescription tells us which boundary conditions to impose. This prescription also follows if we define  $\Delta_F$  by analytically continuing the Euclidean space Green function to Minkowski space. To continue from Minkowski space to Euclidean space, one rotates the  $p^0$  integration contour to the imaginary axis in the counterclockwise direction (to avoid the poles at  $\pm(\omega - i\epsilon)$ ). In order to preserve convergence at infinity,  $x^0$  must be simultaneously rotated in the opposite (clockwise) direction. Changing variables  $p^0 = -ip^4$  and calling  $x^0 = -ix^4$  (positive  $p^0$  rotates to negative  $p^4$  but positive  $x^0$  rotates to positive  $x^4$ ) gives the Euclidean Green function

$$\Delta_F \rightarrow \Delta_E \equiv \int \frac{d^4p}{(2\pi)^4} e^{i(\mathbf{x} \cdot \mathbf{p} + x^4 p^4)} \frac{1}{\mathbf{p}^2 + (p^4)^2 + m^2}.$$

Clearly  $\Delta_E$  satisfies

$$(-\partial_4^2 - \nabla^2 + m^2)\Delta_E(x - y) = \delta^4(x - y).$$

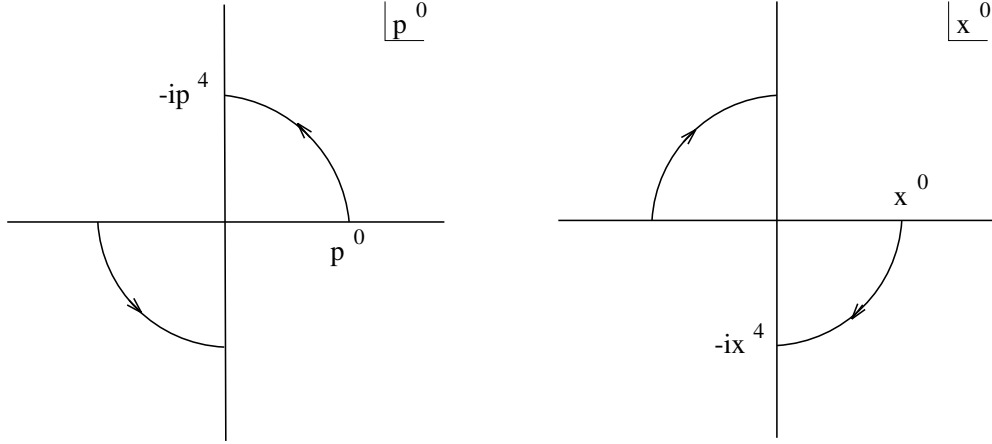


FIG. 1. Wick Rotations.

Next we turn to the evaluation of the propagator for the Dirac field. Remembering that  $\psi$  is fermionic we have

$$\begin{aligned}
 S_F(x_1 - x_2)_{ab} &\equiv \langle 0 | T[\psi_a(x_1)\bar{\psi}_b(x_2)] | 0 \rangle \\
 &= \theta(t_1 - t_2) \langle 0 | \psi_a(x_1)\bar{\psi}_b(x_2) | 0 \rangle - \theta(t_2 - t_1) \langle 0 | \bar{\psi}_b(x_2)\psi_a(x_1) | 0 \rangle \\
 &= \int \frac{d^3 p}{(2\pi)^3 2\omega(\mathbf{p})} [\theta(t_1 - t_2) e^{ip \cdot (x_1 - x_2)} \sum_{\lambda} u_{\lambda}^a(\mathbf{p}) \bar{u}_{\lambda}^b(\mathbf{p}) \\
 &\quad - \theta(t_2 - t_1) e^{ip \cdot (x_2 - x_1)} \sum_{\lambda} v_{\lambda}^a(\mathbf{p}) \bar{v}_{\lambda}^b(\mathbf{p})], \tag{4.10}
 \end{aligned}$$

where in the integrand we have  $p^0 = \omega(\mathbf{p})$ .

To simplify the expression for  $S_F$  we need to evaluate  $\sum_{\lambda} u_{\lambda}^a(\mathbf{p}) \bar{u}_{\lambda}^b(\mathbf{p})$  and  $\sum_{\lambda} v_{\lambda}^a(\mathbf{p}) \bar{v}_{\lambda}^b(\mathbf{p})$ . If we regard them as matrices with indices  $a, b$ , we know we can write each as a linear combination of the 16 matrices  $\Gamma$  we used in constructing the bilinears. By virtue of the sum over helicity, conjugation by the Lorentz transformation matrices simply does the corresponding Lorentz transformation on  $p^{\mu} = (\mathbf{p}, \omega)$ . Thus they must be scalars formed from  $p$  and the matrices  $\Gamma$ . The only possibilities are  $I$  and  $p \cdot \gamma$ , so

$$\sum_{\lambda} u_{\lambda}^a(\mathbf{p}) \bar{u}_{\lambda}^b(\mathbf{p}) = A\delta_{ab} + Bp \cdot \gamma_{ab},$$

and we only need to determine  $A, B$ . First notice that multiplying by the matrix  $m + p \cdot \gamma$  must give 0, which determines  $A = -mB$ . Then, multiplying both sides by  $\gamma^0$  gives  $2\omega \sum_{\lambda} B = -4\omega$  or  $B = -1$ . Thus

$$\sum_{\lambda} u_{\lambda}^a(\mathbf{p}) \bar{u}_{\lambda}^b(\mathbf{p}) = (m - p \cdot \gamma)_{ab}.$$

The definition  $v = i\gamma^2 u^*$  then determines

$$\sum_{\lambda} v_{\lambda}^{\alpha}(\mathbf{p}) \bar{v}_{\lambda}^{\beta}(\mathbf{p}) = [i\gamma^2(m - \mathbf{p} \cdot \gamma^*)(-i\gamma^2)]_{ab} = -(m + \mathbf{p} \cdot \gamma)_{ab}.$$

Inserting these relations into (4.10) yields

$$S_F(x_1 - x_2)_{ab} = \int \frac{d^3 p}{(2\pi)^3 2\omega(\mathbf{p})} [\theta(t_1 - t_2) e^{ip \cdot (x_1 - x_2)} (m - \mathbf{p} \cdot \gamma)_{ab} + \theta(t_2 - t_1) e^{ip \cdot (x_2 - x_1)} (m + \mathbf{p} \cdot \gamma)_{ab}].$$

The final step is to employ the integral representation for the step functions as we did for the scalar propagator. This process results in a four dimensional momentum integral with  $p^0$  substituted for  $\omega$  in the exponents but not in front of  $\gamma^0$ . Then the two terms involving  $m$  and those involving the spatial  $\gamma^k$  combine, after the change of variable  $p \rightarrow -p$  in the second term, exactly as in the scalar case. The two terms involving  $\gamma^0$  have a factor of  $\omega$  which cancels that in the denominator, but then they combine with the opposite relative sign to produce a  $2p^0$  in the numerator. Thus the net result is simply

$$S_F(x_1 - x_2)_{ab} = -i \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot (x_1 - x_2)} \left( \frac{m - \mathbf{p} \cdot \gamma}{m^2 + p^2 - i\epsilon} \right)_{ab}. \quad (4.11)$$

Just as with the scalar propagator  $S_F$  may be recognized as a Green function for the differential dirac operator  $\frac{1}{i}\gamma \cdot \partial + m$ :

$$\left( \frac{1}{i}\gamma \cdot \partial + m \right) S_F(x - y) = -i\delta^4(x - y),$$

with boundary condition dictated by the  $i\epsilon$  prescription. As before this boundary condition can be enforced by defining  $S_F$  as the continuation of the Euclidean Green function  $S_E$ . The continuation from  $S_F$  to  $S_E$  proceeds by rotating the  $p^0$  integration contour to the imaginary axis in the counterclockwise direction (of course rotating  $x^0$  in the opposite direction), changing variables  $p^0 = -ip^4$ , and defining  $x^0 = -ix^4$ ,  $\gamma^0 = -i\gamma_E^4$ :

$$S_F(x - y) \rightarrow S_E(x - y) = \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot (x_1 - x_2)} \left( \frac{m - \mathbf{p} \cdot \gamma_E}{m^2 + p^2} \right)_{ab}. \quad (4.12)$$

$S_E$  satisfies the equation

$$\left( \frac{1}{i}\gamma \cdot \partial + m \right) S_E(x - y) = \delta^4(x - y).$$

#### 4.9. VACUUM EXPECTATIONS FROM LARGE TIME LIMITS OF GENERAL TRANSITION AMPLITUDES.

One obstacle to formulating an efficient perturbation theory for systems with interacting quantum fields is that one can't "turn off" the interactions at early and late times as is possible with externally applied fields. Thus *out* and *in* states are eigenstates of complicated interacting Hamiltonians. One approach to this difficulty is to artificially make the coupling constants time dependent and force them to vanish at early and late times. Another approach, which we shall favor, is to relax the requirement that the initial and final states be eigenstates of the Hamiltonian with vanishing external fields. Then one calculates in first instance a quantity that is not of immediate interest, but which can be simply related to such quantities.

A quantity of more or less direct physical interest is the vacuum expectation value of the time ordered product of several quantum fields. More generally the *outin* matrix element of such a time ordered product is relevant if time varying external fields are present. So let us consider how to obtain this quantity in perturbation theory by first calculating with general initial and final states. Using the evolution operator and assuming  $t_1 > t_2 > \dots > t_n$ , we therefore consider

$$\begin{aligned} \langle f|U(\infty, -\infty)T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|i\rangle = \\ \langle f|U(\infty, t_1)\Omega_{S_1}U(t_1, t_2)\cdots U(t_{n-1}, t_n)\Omega_{S_n}U(t_n, -\infty)|i\rangle. \end{aligned}$$

Choose the time  $T$  so that all external fields vanish for times earlier than  $-T$  and later than  $T$ . Then

$$\begin{aligned} U(t_n, -\infty)|i\rangle &= U(t_n, -T)e^{-i(\infty-T)H_S}|i\rangle \\ &= U(t_n, -T)e^{-i(\infty-T)E_G}\sum_r e^{-i(\infty-T)(E_r-E_G)}|r\rangle\langle r|i\rangle. \end{aligned}$$

We would now like to argue that the infinite oscillations wash out all contributions but the (assumed nondegenerate<sup>\*</sup>) ground state. In a field theory this is quite plausible since the excited states correspond to particles so the sum over  $r$  is really an integral over a range of continuous energies. But even without this smearing, we can make the washing out rigorous by calculating with imaginary time:  $it = \beta > 0$ . Then  $i\infty$  is really  $+\infty$  and all excited states are damped exponentially. Massless particle states could introduce a subtlety here, but the part of phase space that is not exponentially damped is infinitesimal: this has the effect of changing exponential damping to a power law damping. If we buy this argument, then we can assert quite generally that  $U(t_n, -\infty)|i\rangle = U(t_n, -\infty)|0\rangle\langle 0|i\rangle$  and similarly  $\langle f|U(\infty, t_1) = \langle f|0\rangle\langle 0|U(\infty, t_1)$ .

Since we take (as usual) the Heisenberg and Schrödinger pictures to coincide at  $t = -\infty$ , then  $|in\rangle = |0\rangle$  and  $\langle out| = \langle 0|U(\infty, -\infty)$ . Thus we have obtained the relation

$$\begin{aligned} \langle f|U(\infty, -\infty)T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|i\rangle = \\ \langle f|0\rangle\langle 0|i\rangle\langle out|T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|in\rangle. \end{aligned}$$

In other words calculating with any initial and final states that have finite overlap<sup>†</sup> with the true ground state gives us a constant times the desired matrix element. We can easily evaluate the multiplicative constant by considering by the same reasoning

$$\begin{aligned} \langle f|U(\infty, -\infty)|i\rangle &= \langle f|0\rangle\langle 0|i\rangle\langle out|in\rangle \\ &\rightarrow e^{-2i\infty E_G}\langle f|0\rangle\langle 0|i\rangle \quad \text{External Fields} = 0. \end{aligned}$$

Putting this into our relation we obtain

$$\begin{aligned} \langle out|T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|in\rangle = \\ e^{-2i\infty E_G}\frac{\langle f|U(\infty, -\infty)T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|i\rangle}{\langle f|U(\infty, -\infty)_{Ext=0}|i\rangle}, \end{aligned} \quad (4.13)$$

where the subscript on  $U$  in the denominator denotes vanishing external fields. In field theory applications  $E_G$  is the energy of the vacuum, which is zero if we measure all energies relative to that of the vacuum. In the

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\* There are interesting cases of degenerate vacua, when there is “spontaneous symmetry breakdown.” In such cases the choice of initial and final states determines which of the degenerate vacua is picked out.

† The infinite number of degrees of freedom in quantum field theory requires care here: the overlap between different states in a theory with  $n$  degrees of freedom is typically  $f^n$  with  $f < 1$ . Since  $n = \infty$ , we should expect  $\langle f|0\rangle\langle 0|i\rangle \sim e^{-\infty}$ . In field theory  $n = \infty$  because the volume of space is infinite and because space is continuous. Thus strict application of the above relation should be done in the presence of both an infrared and ultraviolet cutoff, which can then be removed after extracting the desired amplitude.

absence of gravity all physical quantities depend only on energy differences, so we lose nothing by doing this. Gravity couples directly to the energy density and therefore is sensitive to the energy as opposed to energy differences, but then  $E_G$  only appears in the combination  $\Lambda \equiv E_G + \Lambda_0$ , with  $\Lambda_0$  the “bare” cosmological constant. Replacing  $\Lambda_0$  by  $\Lambda$  in effect sets  $E_G = 0$ .

The formula (4.13) is a convenient starting point for developing perturbation theory. Any breakup

$$H_S(t) = H_0(t) + H'(t)$$

determines an interaction picture defined by

$$\Omega_I(t) = U_0^{-1}(t, -\infty)\Omega_S U_0(t, -\infty) = U_I(t, -\infty)\Omega(t)U_I^{-1}(t, -\infty),$$

where

$$\begin{aligned} i\dot{U} &= H_S(t)U = UH(t) \\ i\dot{U}_0 &= U_0 H_{0I}(t) \\ i\dot{U}_I &= H_I(t)U_I \end{aligned}$$

and all  $U$ 's are the identity at  $t = -\infty$ . Then the evolution operator satisfies

$$U(t_1, t_2) = U(t_1, -\infty)U^{-1}(t_2, -\infty) = U_0(t_1, -\infty)U_I(t_1, t_2)U_0^{-1}(t_2, -\infty).$$

Plugging these relations into (4.13) then gives

$$\begin{aligned} \langle out|T[\Omega_1(t_1) \cdots \Omega_n(t_n)]|in\rangle = \\ e^{-2i\infty E_G} \frac{\langle f|U_0(\infty, -\infty)T[U_I(\infty, -\infty)\Omega_{1I}(t_1) \cdots \Omega_{nI}(t_n)]|i\rangle}{\langle f|U_0(\infty, -\infty)_{E_{xt=0}}U_I(\infty, -\infty)_{E_{xt=0}}|i\rangle}. \end{aligned} \quad (4.14)$$

This formula is completely general: we have even allowed  $H_0$  to contain time varying external fields, which is hardly ever done in practice. Since all operators in this formula are in interaction picture, it is most convenient to choose  $|i\rangle, |f\rangle$  to have simple properties with respect to  $H_{0I}(-\infty)$ . Let us call the ground state of this operator  $|in, 0\rangle$ . Then  $\langle in, 0|U_0(\infty, -\infty)$  is the ground state of  $H_{0I}(+\infty)$  and therefore deserves the name  $\langle out, 0|$ . When all external fields vanish,  $H_{0I}$  is time independent and we call its ground state  $|0, I\rangle \equiv |in, 0\rangle$  and its ground state energy  $E_0$ . Then  $\langle in, 0|U_0(\infty, -\infty)_{E_{xt=0}} = e^{-2i\infty E_0} \langle 0, I|$ . Thus choosing  $|i\rangle = |f\rangle = |0, I\rangle = |in, 0\rangle$  we obtain the useful formula

$$\begin{aligned} \langle out|T[\Omega_1(t_1) \cdots \Omega_n(t_n)]|in\rangle = \\ e^{-2i\infty(E_G - E_0)} \frac{\langle out, 0|T[U_I(\infty, -\infty)\Omega_{1I}(t_1) \cdots \Omega_{nI}(t_n)]|in, 0\rangle}{\langle 0, I|U_I(\infty, -\infty)_{E_{xt=0}}|0, I\rangle}. \end{aligned} \quad (4.15)$$

In the usual case where we do not include external fields in  $H_0$ , the formula simplifies further

$$\begin{aligned} \langle out|T[\Omega_1(t_1) \cdots \Omega_n(t_n)]|in\rangle = e^{-2i\infty E_G} \langle G|T[\Omega_1(t_1) \cdots \Omega_n(t_n)]|G\rangle = \\ e^{-2i\infty E_G} \frac{\langle 0, I|T[U_I(\infty, -\infty)\Omega_{1I}(t_1) \cdots \Omega_{nI}(t_n)]|0, I\rangle}{\langle 0, I|U_I(\infty, -\infty)_{E_{xt=0}}|0, I\rangle}. \end{aligned} \quad (4.16)$$

Using the Wick expansion one can describe the perturbation series for the numerators and denominators of these formulas using Feynman diagrams. The diagrams contributing to the denominator are all those completely disconnected from either external fields or from the points assigned to the operators in the numerator. The numerator contains this same sum of diagrams as a multiplicative factor. Thus the division by the denominator is achieved by simply deleting all such disconnected “vacuum” diagrams from the expansion of the numerator.

## 5. Perturbation Theory for $\phi^3$ Scalar Field Theory

Let us now apply what we have learned to the scalar field theory with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{m_0^2}{2}\phi^2 - \frac{g}{3!}\phi^3$$

One's first thought is to take the perturbation in interaction picture to be  $H'_I(t) = (g/3!) \int d^3x \phi^3(x, t)$ . We shall find that, because the interactions don't turn off at early and late times, this is too glib. But let's see where it leads. We have no external fields, so the time dependent perturbation theory reads

$$\langle G|T[\phi(x_1)\cdots\phi(x_n)]|G\rangle = \frac{\langle 0, I|T[\exp\{-\frac{ig}{3!}\int d^4x\phi_I^3(x)\}\phi_I(x_1)\cdots\phi_I(x_n)]|0, I\rangle}{\langle 0, I|T\exp\{-\frac{ig}{3!}\int d^4x\phi_I^3(x)\}|0, I\rangle}. \quad (5.1)$$

Her  $|G\rangle$  is the exact ground state.

### 5.1. THE VACUUM

The simplest application of the dyson formula is the case without any fields in the time ordered product:

$$\frac{\langle G|I|G\rangle = \langle G|G\rangle = T[\exp\{-\frac{ig}{3!}\int d^4x\phi_I^3(x)\}]|0, I\rangle}{\langle 0, I|T\exp\{-\frac{ig}{3!}\int d^4x\phi_I^3(x)\}|0, I\rangle} = 1. \quad (5.2)$$

Which is consistent but pretty trivial. The numerator and denominator are separately complicated expressions but complications all cancel. Let's look more closely at the structure of the numerator:

$$\text{Num} = 1 + \frac{1}{2} \frac{(-ig)^2}{6^2} \int d^4x d^4y \langle 0, I|T\phi_I^3(x)\phi_I^3(y)|0, I\rangle.$$

Applying the Wick expansion to the second term can be visualized as two Feynman diagrams. These diagrams are characterized by having no external lines and are sometimes called vacuum bubbles. The bottom line for them is that they are always cancelled, so we never have to deal with them.

### 5.2. ONE-POINT FUNCTION

The next simplest thing to consider is the VEV of the scalar field,  $\langle G|\phi(x)|G\rangle = \langle G|\phi(0)|G\rangle$  by translation invariance. It is just a constant number, call it  $v$ . What if we start trying to calculate it in perturbation theory. Applying the Dyson formula to first order gives

$$v = \langle G|\phi(0)|G\rangle \approx \frac{-ig}{3!} \int d^4x \langle G|T\phi(0)\phi^3(x)|G\rangle + \cdots$$

The Wick expansion shows this as a ‘‘tadpole’’ diagram. The actual value of the number  $v$  is not particularly significant. We chose  $H'$  so that at zeroth order in perturbation theory  $\langle G|\phi(0)|G\rangle_0 = 0$ . But then the perturbation makes it non zero. As we continue perturbation theory, it is much more convenient to write  $\phi = v + \hat{\phi}$  where  $\langle G|\hat{\phi}(0)|G\rangle_0 = 0$  exactly to all orders in perturbation theory. In terms of  $\hat{\phi}$  the Lagrangian reads

$$\mathcal{L} = -\frac{1}{2}(\partial\hat{\phi})^2 - \frac{m_0^2}{2}(v + \hat{\phi})^2 - \frac{g}{3!}(v + \hat{\phi})^3 = -\frac{1}{2}(\partial\hat{\phi})^2 - \frac{m^2}{2}\hat{\phi}^2 - \left[-\frac{m^2}{2}\hat{\phi}^2 + \frac{m_0^2}{2}(v + \hat{\phi})^2 + \frac{g}{3!}(v + \hat{\phi})^3\right]$$

where  $m$  represents the true physical mass of the scalar particle, in general different from the ‘‘bare’’ input mass  $m_0$ . The correct procedure is to take all the terms we have enclosed in square brackets to define the

perturbation  $H'$ .

$$\mathcal{H}' = -\frac{m^2}{2}\hat{\phi}^2 + \frac{m_0^2}{2}(v + \hat{\phi})^2 + \frac{g}{3!}(v + \hat{\phi})^3 \equiv \frac{g}{3!}\hat{\phi}^3 + \frac{\delta_m}{2}\hat{\phi}^2 + \delta_v\hat{\phi} + \frac{m_0^2}{2}v^2 + \frac{g}{3!}v^3$$

The last two terms are constants which will cancel between numerator and denominator. The values of  $\delta_m$  and  $\delta_v$  are chosen at the end of the calculation so that  $\langle G | \hat{\phi} | G \rangle = 0$  and so that  $m$  stays the physical mass. In practice we never need to find  $\delta_v$  because its only effect is to set to zero all ‘‘tadpole’’ diagrams and subdiagrams. Thus all we have to do is delete all tadpoles and then forget about  $\delta_v$ .

### 5.3. TWO POINT FUNCTION AND THE PHYSICAL MASS

We have seen that the Fourier transform of the free field two point function has a pole at  $p^2 = -m^2$ . We now argue that this feature holds generally for two point functions even in the interacting case.

Let’s first see how this happens in perturbation theory to second order

$$\int d^4x e^{-iq \cdot x} \langle G | T \phi(x) \phi(0) | G \rangle = \frac{-i}{q^2 + m_0^2} + \frac{1}{2} \frac{(-ig)^2}{3!^2} \int d^4x e^{-iq \cdot x} \langle 0, I | T \phi_I(x) \phi_I(0) \int d^4y d^4z \phi_I^3(y) \phi_I^3(z) | 0, I \rangle$$

In the second term, if both  $\phi$ ’s hit the same Hamiltonian we get a tadpole correction to the propagator which is simply proportional to  $(q^2 + m_0^2)^{-2}$ . If each  $\phi$  hits a different Hamiltonian we are left with

$$\begin{aligned} & \frac{-i}{q^2 + m_0^2} + \frac{(-ig)^2}{2} \left( \frac{-i}{q^2 + m_0^2} \right)^2 \int d^4y e^{-iq \cdot (y-z)} \langle 0, I | T \phi_I(y-z) \phi_I(0) | 0, I \rangle^2 \\ & \equiv \frac{-i}{q^2 + m_0^2} \left[ 1 + \frac{-i}{q^2 + m_0^2} (-i) \Pi(q^2) \right] \approx \frac{-i}{q^2 + m_0^2 + \Pi(q^2)} \end{aligned}$$

We see that the pole location has shifted to  $q^2 = -m^2$  where  $m$  satisfies

$$-m^2 + m_0^2 + \Pi(-m^2) = 0.$$

We now see that it would be a good idea to use the true mass  $m$  in the zeroth order propagator and tune the  $\delta_m$  counterterm so that corrections don’t change the pole location, i.e. so that  $\Pi(-m^2) = 0$  to all orders in perturbation theory.

Now, we give the general argument:

$$\int d^4x e^{-iq \cdot x} \langle G | T \phi(x) \phi(0) | G \rangle$$

We lose no generality in taking one point to be 0 because of translation invariance. Fix  $t > 0$  and insert a complete set of energy momentum eigenstates between the two fields:

$$\begin{aligned} \int_0^\infty dt \int d^3x e^{-iq \cdot x} \sum_n \int d^3p \langle G | \phi(x) | n, \vec{p} \rangle \langle n, \vec{p} | \phi(0) | G \rangle &= \int_0^\infty dt \int d^3x e^{-iq \cdot x} \sum_n \int d^3p e^{ip_n \cdot x} |\langle n, \vec{p} | \phi(0) | G \rangle|^2 \\ &= \sum_n (2\pi)^3 \int_0^\infty dt e^{i(q_0 - E_n)t} |\langle n, \vec{q} | \phi(0) | G \rangle|^2 \\ &= \sum_n \frac{-(2\pi)^3}{i(q_0 - E_n + i\epsilon)} |\langle n, \vec{q} | \phi(0) | G \rangle|^2 \end{aligned}$$

If  $|n, \vec{q}\rangle$  is a single particle state,  $E_n = \sqrt{\vec{q}^2 + m_n^2}$  and the pole at  $q^2 = -m_n^2$  is explicit. If it is a multi-particle state the singularity is smeared and becomes a branch point. The residue of the pole depends on the

matrix element  $\langle n, \vec{q} | \phi(0) | G \rangle \equiv \sqrt{Z} / [(2\pi)^{3/2} \sqrt{\omega(\vec{q})}]$  for a single particle state by Lorentz invariance.  $Z < 1$  will not be unity as in the free particle case and the single particle pole is

$$\frac{-i}{p^2 + m_0^2 - i\epsilon} \rightarrow \frac{-iZ}{p^2 + m^2 - i\epsilon}$$

This change in residue  $Z$  is called “wave-function” renormalization.

The argument we have just given shows that  $\phi(x) | g \rangle$  has a single particle component, as well as multi-particle components. It would be nice to use  $\phi(x)$  as a creation operator, but since it also creates multi-particle states, we need somehow arrange these unwanted components to disappear. Taking the spatial Fourier transform  $\int d^3x e^{i\vec{q}\cdot\vec{x}} \phi(x)$  only guarantees that it creates momentum  $\vec{q}$  but not that it creates only a single particle state. If we could select the component with frequency  $e^{i\omega(\vec{q})t}$ , with  $\omega(\vec{q}) = \sqrt{\vec{q}^2 + m^2}$ . A temporal Fourier transform  $\int_{-T}^T dt e^{-i\omega(\vec{q})t} \int d^3x e^{i\vec{q}\cdot\vec{x}} \phi(x)$  would do this to accuracy  $1/2T$ . Since the single particle energy is separated by a gap  $O(m)$  from the vacuum and multi-particle energies this is good enough as long as  $2mT \ll 1$ . In summary

$$\frac{1}{2T} \int_{t_0-T}^{t_0+T} dt e^{-i\omega(\vec{q})t} \int \frac{d^3x}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \phi(x) | G \rangle \equiv \int F(t-t_0) dt e^{-i\omega(\vec{q})t} \int \frac{d^3x}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \phi(x) | G \rangle \equiv A^\dagger(t_0, \vec{q}) | G \rangle$$

is a pure single particle state, with mass  $m$  and momentum  $\vec{q}$ , of the exact interacting theory. Taking out the factors as shown gives  $A(t_0)$  the normalization

$$\langle G | A(t_0, \vec{q}') A^\dagger(t_0, \vec{q}) | G \rangle \approx \int d^3p \langle G | A(t_0, \vec{q}') | \vec{p} \rangle \langle \vec{p} | A^\dagger(t_0, \vec{q}) | G \rangle \approx \delta(\vec{q}' - \vec{q}) |\langle \vec{q} | \phi(0) | G \rangle|^2$$

It is constructed from a Heisenberg picture field operator smeared in the neighborhood of time  $t_0$ .

We can make a two particle state by applying a second  $A^\dagger$ . However, it only has a clean two particle interpretation when the two particles are in wave packets space-like separated from each other. In that case the two  $A^\dagger$ 's commute and represent two independent particles. However, if they are aimed to scatter, the packets will eventually overlap and scattering can occur.

**Reduction Formulae** Let  $-\infty < t_1, t_2, \dots, t_n < \infty$ . Call  $f(\vec{x}, t) = \int d^3q f(\vec{q}) e^{i\vec{x}\cdot\vec{q} - i\omega(\vec{q})t} / (2\pi)^3$ . Then it is easy to show

$$\begin{aligned} & T \{ A_a(\infty) \Omega_1(x_1) \dots \Omega_n(x_n) \} - T \{ \Omega_1(x_1) \dots \Omega_n(x_n) A_a(-\infty) \} \\ &= \int_{-\infty}^{\infty} dt_0 \frac{d}{dt_0} T \{ A_a(t_0) \Omega_1(x_1) \dots \Omega_n(x_n) \} \\ &= \int_{-\infty}^{\infty} dt_0 \int d^4x F(t-t_0) \frac{\partial}{\partial t} f_a^*(\mathbf{x}, t) T \{ \phi(x) \Omega_1(x_1) \dots \Omega_n(x_n) \} \\ &= \int_{-\infty}^{\infty} dt_0 \int d^4x F(t-t_0) f_a^*(\mathbf{x}, t) \left( i\omega_a(-i\nabla) + \frac{\partial}{\partial t} \right) T \{ \phi(x) \Omega_1(x_1) \dots \Omega_n(x_n) \} \\ & T \{ A_a^\dagger(\infty) \Omega_1(x_1) \dots \Omega_n(x_n) \} - T \{ \Omega_1(x_1) \dots \Omega_n(x_n) A_a^\dagger(-\infty) \} \\ &= \int_{-\infty}^{\infty} dt_0 \int d^4x F(t-t_0) \frac{\partial}{\partial t} f_a(\mathbf{x}, t) T \{ \phi(x) \Omega_1(x_1) \dots \Omega_n(x_n) \} \\ &= \int_{-\infty}^{\infty} dt_0 \int d^4x F(t-t_0) f_a(\mathbf{x}, t) \left( -i\omega_a(i\nabla) + \frac{\partial}{\partial t} \right) T \{ \phi(x) \Omega_1(x_1) \dots \Omega_n(x_n) \}. \end{aligned}$$

In these reduction formulae we have used the fact that by construction  $f_a$  satisfies a Schrödinger equation with hamiltonian  $\omega_a(-i\nabla) = \sqrt{m_a^2 - \nabla^2}$ . Spatial integration by parts then allows the spatial derivatives to be transferred to  $\phi(x)$ .

The reduction formulas can be used to relate scattering amplitudes represented as, e.g.

$$\langle G| A(\infty, \vec{q}_1) A(\infty, \vec{q}_2) A^\dagger(-\infty, \vec{p}_1) A^\dagger(-\infty, \vec{p}_2) |G\rangle$$

to Fourier transforms of

$$\langle G| T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) |G\rangle \equiv \int \prod_k \frac{d^4 q_k}{(2\pi)^4} \exp \left\{ i \sum_k q_k \cdot x_k \right\} \mathcal{T}(q_1, q_2, q_3, q_4)$$

The scattering amplitude is proportional to  $\mathcal{T}$  in the on-shell limit  $q_k^2 \rightarrow -m_k^2$ . Then applying  $\partial/\partial t_k \pm i\omega(\mp i\nabla_k)$  is equivalent to multiplying  $\mathcal{T}$  by  $-iq^0 \pm i\omega(\vec{q})$ , which vanishes in one of the on-shell limits when  $q^0 \rightarrow \pm\omega(\vec{q})$ . There will be one such on-shell zero for each external particle. Thus there is only scattering if the F.T. has a mass-shell pole for each particle. The reduction formula gives the precise connection including all normalization factors. But it is quicker to get all the right factors by just considering Feynman diagrams. The required poles come from complete propagators attached to each external line. Lorentz invariance determines all factors up to overall  $Z$  factors. Therefore we can use lowest order perturbation theory to determine everything. The rule is to “amputate” all external legs and replace each propagator with  $\sqrt{Z}e^{\pm iq \cdot x}/(2\pi)^{3/2}/\sqrt{2\omega}$ .

In summary, by choosing the perturbation carefully we set things up so that  $\langle G| \hat{\phi} |G\rangle = 0$  to all orders in perturbation theory (this just amounts to dropping all tadpole corrections!), and so that  $\Pi(-m^2) = 0$  to all orders in perturbation theory. Then near mass shell  $\Pi(q^2) \approx (q^2 + m^2)(Z^{-1} - 1)$ , and we see how the residue shifts. Since we don't know  $m^2$  and  $v$  in advance we have to keep adjusting  $\delta_m$  and  $\delta_v$  as we proceed.

#### 5.4. FEYNMAN DIAGRAMS

The Wick expansion is very efficiently analyzed via Feynman diagrams. These are a collections of lines, one for each propagator in the Wick expansion, connected together at vertices determined by the interaction Lagrangian. In the case at hand,  $\mathcal{L}' = -\frac{g}{3!}\phi^3 - \frac{\delta_m}{2}\phi^2 - \delta_v\phi$ .

- 1 List Feynman rules in  $x$  space for  $\phi^3$ .
- 2 Motivate and translate rules to  $p$  space.
- 3 Connectedness: sufficient to focus on connected diagram– general term is product of connected pieces.
- 4 Example: Sum of all vacuum graphs = exp of sum of all connected vacuum graphs.
- 5 Lowest order four point function.

#### 5.5. CROSS SECTIONS AND RATES

Write the scattering amplitude as

$$(2\pi)^4 \delta(P_f - p_1 - p_2) \mathcal{T} \equiv (2\pi)^4 \delta(P_f - p_1 - p_2) \frac{\mathcal{M}}{(2\pi)^3 \sqrt{4\omega_1\omega_2} \prod_f [(2\pi)^{3/2} \sqrt{2\omega_f}]}$$

For a scattering process the initial state is two separated wave packets aimed at each other so they will meet and scatter. This situation can be arranged by a momentum space wave function  $f(\vec{p}_1, \vec{p}_2)$ , narrowly peaked

about  $\vec{p}_i = \hat{z}p_i$ . For the lab frame  $p_2 = 0$ , and for the CM frame  $p_2 = -p_1$ . Then

$$\text{Amplitude} = \int d^3p_1 d^3p_2 (2\pi)^4 \delta(P_f - p_1 - p_2) \mathcal{T} f(\vec{p}_1, \vec{p}_2) \approx (2\pi)^4 \mathcal{T} \int d^3p_1 \delta(E_f - \omega_1 - \omega_2) f(\vec{p}_1, \vec{P}_f - \vec{p}_1)$$

To deal with the last delta function write  $d^3p_1 = d^2p_1^\perp dp_1^z = d^2p_1^\perp d(\omega_1 + \omega_2) \frac{1}{v_{12}}$  where  $v_{12} = |\partial\omega_1/\partial p_1^z + \partial\omega_2/\partial p_1^z| = |v_1 - v_2|$  is the relative velocity of the two initial packets. Then

$$\text{Amplitude} \approx \frac{(2\pi)^4}{v_{12}} \mathcal{T} \int d^2p_1 f(\vec{p}_1, \vec{P}_f - \vec{p}_1)$$

This function is very sharply peaked about  $P_f = p_1^0 + p_2^0$  with width given by the initial packet. Its square will also be sharply peaked, and safely approximated by a delta function:

$$\text{Amp}^2 \approx \frac{(2\pi)^8}{v_{12}^2} |\mathcal{T}|^2 \delta(P_f - p_1^0 - p_2^0) \int d^4P \left| \int d^2p_1 f(\vec{p}_1, \vec{P} - \vec{p}_1) \right|^2$$

Now consider

$$\psi(x, y, p_1^z, \vec{P}) = \int \frac{d^2p}{2\pi} e^{ixp^x + iyp^y} f(\vec{p}_1, \vec{P} - \vec{p}_1)$$

It's square is the probability per  $dp_1^z d^3P$  that the relative transverse displacement of the incident particles is in  $dx dy$  of the point  $(x, y)$ . This quantity appears in our formula for the case  $x = y = 0$  and integrated over  $d^3P dE_f \approx v_{12} d^3P dp_1^z$ .  $\int d^4P \left| \int d^2p_1 f(\vec{p}_1, \vec{P} - \vec{p}_1) \right|^2 \approx (2\pi)^2 v_{12} \times$  the total probability the incident particles are within  $dx dy$  of  $x = y = 0$ . The coefficient of this probability is just what we mean by the differential cross section

$$d\sigma = \prod_f d^3p_f \frac{(2\pi)^{10}}{v_{12}} \delta(P_f - p_1 - p_2) |\mathcal{T}|^2 = \prod_f \frac{d^3p_f}{(2\pi)^3 2\omega_f} \frac{1}{4\omega_1 \omega_2 v_{12}} (2\pi)^4 \delta(P_f - p_1 - p_2) |\mathcal{M}|^2$$

The last form is the one that is most convenient to remember. In it  $\mathcal{M}$  has all external wave function factors  $1/(2\pi)^{3/2}/\sqrt{2\omega}$  removed. It is the Fourier transform of the appropriate time ordered product, with the propagator on each external line amputated and replaced with the appropriate  $\sqrt{Z}$  factor.

An apparently simpler process than scattering is the decay of a single particle into any final state. In this case we write the decay probability amplitude as

$$(2\pi)^4 \delta(P_f - p_1) \mathcal{T} \equiv (2\pi)^4 \delta(P_f - p_1) \frac{\mathcal{M}}{(2\pi)^{3/2} \sqrt{2\omega_1} \prod_f [(2\pi)^{3/2} \sqrt{2\omega_f}]}$$

However in this case putting the initial particle in a wave packet  $f(\vec{p})$  only takes care of the spatial delta function, and after this we still can't meaningfully square the amplitude to get a probability. The problem is that the formula assumes that the initial particle has existed for an infinite time, which is not consistent with the fact that it can decay. It has a finite lifetimes  $\tau$ , and it can only be approximately be described as a particle with definite mass for a time  $2T \ll \tau$ . The correct way to deal with unstable systems involves including their production as well as decay. To deal with the application of the above formula, we can replace the energy conserving delta function with a finite time version

$$2\pi \delta(E_f - \omega_1) \rightarrow \int_{-T}^T dt e^{-it(E_f - \omega_1)} = \frac{2 \sin(E_f - \omega_1)T}{E_f - \omega_1}$$

where we understand that  $2T \ll \tau$ . Application of the formula implicitly assumes that  $T \gg 1/\Delta E$  where  $\Delta E$  characterizes the scale over which  $\mathcal{T}$  varies. Thus we should only try to apply this formula for the decay

of a particle with lifetime  $\tau \gg 1/\Delta E$ . In this situation we can approximate the square of the finite time delta function with a sharp delta function:

$$\left| \frac{2 \sin(E_f - \omega_1)T}{E_f - \omega_1} \right|^2 \approx \delta(E_f - \omega_1) 4 \int dE \frac{\sin^2 ET}{E^2} = 2\pi \delta(E_f - \omega_1)(2T)$$

With this interpretation we arrive at the following formula for the differential decay probability

$$\begin{aligned} dP &= \prod_f d^3 p_f (2\pi)^7 |\mathcal{T}|^2 \delta(E_f - \omega_1)(2T) |f(\vec{P}_f)|^2 \approx \prod_f d^3 p_f (2\pi)^7 |\mathcal{T}|^2 \delta(P_f - p_1)(2T) \int d^3 p |f(\vec{p})|^2 \\ &= \prod_f d^3 p_f (2\pi)^7 |\mathcal{T}|^2 \delta(P_f - p_1)(2T) = 2T \prod_f \frac{d^3 p_f}{(2\pi)^3 2\omega_f} \frac{1}{2\omega_1} (2\pi)^4 \delta(P_f - p_1) |\mathcal{M}|^2 \end{aligned}$$

We see that in this approximate description the decay probability is proportional to the duration of the process  $2T$ . This linear dependence is clearly only meaningful for  $2T \ll \tau$ . However the probability per unit time, the decay rate, is independent of  $T$ :

$$d\Gamma \equiv \frac{dP}{2T} = \prod_f \frac{d^3 p_f}{(2\pi)^3 2\omega_f} \frac{1}{2\omega_1} (2\pi)^4 \delta(P_f - p_1) |\mathcal{M}|^2.$$

Summing over all allowed final states, including integrating over phase space gives the total decay rate  $\Gamma$ . We can use it to determine the long time behavior of the decay process. Let  $P(t)$  be the probability that the particle has *not* decayed in a time  $t$  after its creation  $P(0) = 1$ . Then

$$\frac{dP}{dt} = -\Gamma P(t), \quad \rightarrow \quad P(t) = e^{-\Gamma t}$$

This provides a precise definition for the lifetime of the particle, namely,  $\tau = 1/\Gamma$ . Also the probability that the particle has decayed in time  $t$  is  $1 - P(t) = 1 - e^{-\Gamma t} = 1 - e^{-t/\tau} \approx t/\tau = t\Gamma$ , when  $t \ll \tau$ , showing the consistency of our rate formula.

As a very useful example, we specialize these formulas to the case of two particle final states. Four of the six phase space integrals can be evaluated using the delta functions.

$$d^3 p_1^f d^3 p_2^f \delta(p_1^f + p_2^f - P_i) = d^3 p_1^f \delta(\omega_1^f(p_1^f) + \omega_2^f(P_i - p_1^f) - E_i) = d\Omega \frac{p_1^{f2} \omega_1^f \omega_2^f}{p_1^f E_i - P_i \omega_1^f \cos \theta}$$

then we have

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{|\mathcal{M}|^2}{64\pi^2 \omega_1 \omega_2 v_{12}} \frac{p_1^{f2}}{p_1^f E_i - P_i \omega_1^f \cos \theta} \\ \frac{d\Gamma}{d\Omega} &= \frac{|\mathcal{M}|^2}{32\pi^2 \omega_1} \frac{p_1^{f2}}{p_1^f \omega_1 - p_1 \omega_1^f \cos \theta} \end{aligned}$$

These formulas dramatically simplify in the center of mass system when  $\vec{P}_i = 0$ . Then  $\omega_1 \omega_2 v_{12} = p_1(\omega_1 + \omega_2)$  and

$$\begin{aligned} \frac{d\sigma}{d\Omega}_{\text{CM}} &= \frac{p_1^f}{p_1} \frac{|\mathcal{M}|^2}{64\pi^2 (\omega_1 + \omega_2)^2} \\ \frac{d\Gamma}{d\Omega}_{\text{CM}} &= \frac{|\mathcal{M}|^2}{32\pi^2 m_1^2} p_1^f \end{aligned}$$

We finally note that when calculating total rates and cross sections with identical particles in the final state, the result of integrating over phase space must be divided by  $\prod_c n_c!$  when there are  $n_c$  identical particles of type  $c$ .

## 6. Path History Quantization

One of the principal drawbacks of the canonical operator formulation of quantum mechanics is that it obscures symmetries that bring in time in an essential way. Lorentz boosts are of this type, so the operator approach inevitably hides the full symmetries of relativity. This is of course also true in the Hamilton equation form of classical mechanics. In classical mechanics one can work with the Lagrangian and Action Principle which keep dynamical symmetries like Poincaré transparent. The path integral approach to quantum mechanics is the quantum analogue of this alternative and, as we shall see, is a much more convenient formulation of quantum field theory than the operator approach. Even in the operator approach we have seen the advantage of expressing results in terms of finite time evolutions, since it is these that reflect the true symmetries of the system. The central object in the path integral approach then is not a state but an amplitude for the evolution of one state into another.

To keep notation simple, we shall suppress indices in dealing with a general quantum mechanical system the coordinates of which are collectively denoted  $q$  and the conjugate momenta of which are  $p$ . Then we seek an alternative scheme for calculating, for example, the amplitude

$$\langle q'' | U(t_1, t_2) | q' \rangle .$$

$U$  is of course a very complicated operator for finite time differences, but as  $t_1 \rightarrow t_2$ , it is simply related to the Hamiltonian.

Thus we are led to break up the time interval into infinitesimal pieces  $t_1 - t_2 = (N + 1)a$  and employ the closure property of  $U$  to write it as a product of  $N + 1$  infinitesimal evolutions.

$$\begin{aligned} \langle q'' | U(t_1, t_2) | q' \rangle &= \langle q'' | U(t_1, t_1 - a) U(t_1 - a, t_1 - 2a) \cdots U(t_2 + a, t_2) | q' \rangle \\ &= \int \prod_{k=1}^N dq_k \langle q'' | U(t_1, t_1 - a) | q_N \rangle \langle q_N | U(t_1 - a, t_1 - 2a) | q_{N-1} \rangle \\ &\quad \cdots \langle q_1 | U(t_2 + a, t_2) | q' \rangle . \end{aligned}$$

Next, we write, assuming  $H_S$  is constant over a time interval  $a$ ,

$$\begin{aligned} &\langle q_k | U(t_2 + ka, t_2 + (k - 1)a) | q_{k-1} \rangle \\ &\approx \int dp_{k-1} \langle q_k | e^{-iaH(t_2+(k-1/2)a)/2} | p_{k-1} \rangle \langle p_{k-1} | e^{-iaH(t_2+(k-1/2)a)/2} | q_{k-1} \rangle \\ &\equiv \int dp_{k-1} \exp\left\{-\frac{i}{2\hbar}a(\mathcal{H}_k(q_k, p_{k-1}, ia) + \mathcal{H}_k(q_{k-1}, p_{k-1}, -ia^*))\right\} \\ &\quad \langle q_k | p_{k-1} \rangle \langle p_{k-1} | q_{k-1} \rangle \\ &= \int \frac{dp_{k-1}}{2\pi\hbar} \exp\left\{\frac{i}{\hbar}[p_{k-1}(q_k - q_{k-1}) \right. \\ &\quad \left. - \frac{a}{2}(\mathcal{H}_k(q_k, p_{k-1}, ia) + \mathcal{H}_k(q_{k-1}, p_{k-1}, -ia^*))]\right\}, \end{aligned}$$

where for the moment  $\mathcal{H}_k$  is defined by these equations. Putting everything together we obtain

$$\begin{aligned} \langle q'' | U(t_1, t_2) | q' \rangle &= \int \prod_{k=1}^N \frac{dq_k dp_k}{2\pi\hbar} \frac{dp_0}{2\pi\hbar} \exp\left\{\frac{i}{\hbar} \sum_{k=1}^{N+1} [p_{k-1}(q_k - q_{k-1}) \right. \\ &\quad \left. - \frac{a}{2}(\mathcal{H}_k(q_k, p_{k-1}, ia) + \mathcal{H}_k(q_{k-1}, p_{k-1}, -ia^*))]\right\}, \end{aligned} \tag{6.1}$$

where  $q_{N+1} \equiv q''$  and  $q_0 \equiv q'$ . Apart from the assumption that external fields are constant over the time interval  $a$  this formula is exact. But it is not useful until we get a simple approximation for  $\mathcal{H}_k$ . For  $a \rightarrow 0$

we should be able to approximate

$$\begin{aligned}
\langle q_k | e^{-iaH(t_2+(k-1/2)a)/2} | p_{k-1} \rangle &\approx \langle q_k | (1 - i\frac{a}{2}H(t_2 + (k-1/2)a)) | p_{k-1} \rangle \\
&\approx (1 - i\frac{a}{2}H_k^W(q_k, p_{k-1})) \langle q_k | p_{k-1} \rangle \\
&\approx \exp\{-i\frac{a}{2}H_k^W(q_k, p_{k-1})\} \langle q_k | p_{k-1} \rangle
\end{aligned}$$

where  $H_k^W(q, p)$  is the operator  $H(t_2 + (k-1/2)a)$  rewritten through use of the canonical commutation relations with all  $p$ 's on the right and all  $q$ 's on the left. After this is done  $q$  can then be replaced with the eigenvalue  $q_k$  and  $p$  with the eigenvalue  $p_{k-1}$ . In the limit  $a \rightarrow 0$  with  $t_1 - t_2 = (N+1)a$  fixed it should be valid to replace  $\mathcal{H}_k(q_k, p_{k-1})$  with  $H_k^W(q_k, p_{k-1})$ . We define the quantity

$$H_k^Q(q_k, q_{k-1}, p_{k-1}) = \frac{1}{2}(H_k^W(q_k, p_{k-1}) + H_k^W(q_{k-1}, p_{k-1})^*) \quad (6.2)$$

which appears in the path integral. Up to the reordering terms  $H_k^W$  is just the classical Hamiltonian for the system. In the common case where the Hamiltonian is a function of  $p$ 's plus a function of  $q$ 's it is nothing more nor less than the classical Hamiltonian. But notice that the object appearing in the path integral is (6.2) which depends on the *two* coordinates describing the initial and final states of the basic unit of propagation even in the case where there are no reordering terms, when it is simply the average of the classical Hamiltonian over the two coordinates. In this continuum limit we can think of the sum in the exponent in (6.1) as an integral

$$\frac{i}{\hbar} \int_{t_2}^{t_1} dt [p(t)\dot{q}(t) - \frac{1}{2}(H^W(q_>(t), p(t), t) + H^W(q_<(t), p(t), t)^*)]. \quad (6.3)$$

The coefficient of  $\frac{i}{\hbar}$  is just the classical action  $\int dt L$  plus terms of order  $\hbar^2$  expressed as a Legendre transform of the Hamiltonian. In the continuum limit, the number of integration variables tends to infinity and the limit gives the definition of the path integral representation of  $\langle q_f | U(t_1, t_2) | q_i \rangle$ .

The Classical Limit For a general Hamiltonian, this is as far as one can go without further approximations. One such approximation one can always try is the limit  $\hbar \rightarrow 0$ , the classical limit. Such a limit is dominated by the functions  $q(t), p(t)$  for which the coefficient of  $i/\hbar$  is stationary. Since, in this limit, this coefficient is just the classical action, the stationarity conditions are simply the classical Hamilton equations:  $\dot{q} = \partial H / \partial p$  and  $\dot{p} = -\partial H / \partial q$ . Thus the path history version of the quantum principle is that for  $\hbar \neq 0$  transition amplitudes are computed by evaluating  $e^{i(Action)/\hbar}$  for all possible histories and averaging this expression over all such histories. The classical limit is understood as the situation in which this average is dominated by solutions of the classical equations of motion.

Imaginary Time. In working with the path integral it is technically advantageous to work with actually damped integrands rather than the oscillating integrand occurring in the quantum path integral. This can be achieved with the Wick rotation  $it = \tau$  where real positive  $t$  is rotated to real positive  $\tau$ . Considering the basic unit of the path integral, the matrix element of the operator,  $e^{-iaH/\hbar}$ , we see that this rotation is mathematically justified when  $H$  is an operator bounded below, *i.e.* its eigenvalue spectrum is bounded below. It obviously should not be attempted if  $H$  has eigenvalues down to  $-\infty$ . Fortunately, most reasonable physical systems have this property, and for these the Wick rotated path integral is the superior one to work with, especially for applications outside of perturbation theory. For a constant Hamiltonian (no external

fields) this path integral calculates  $\langle q_f | e^{-\beta H} | q_i \rangle$  where  $\beta = (\tau_1 - \tau_2)/\hbar$ . If we identify  $q_f = q_i = q$  and integrate over  $q$ , it calculates  $\text{Tr } e^{-\beta H}$ , the statistical mechanical partition function for temperature  $1/\beta$ . In this way the Wick rotated path integral is related to a quantity of direct physical interest in another context. In the limit  $\beta \rightarrow \infty$  (low temperature) one obtains information about the energy levels and degeneracies of the system. However, for applications to quantum mechanics it is necessary to continue back to real time at the end of the calculation of physical transition amplitudes.

Technically the Wick rotation amounts to replacing  $ia$  by  $\delta > 0$  in (6.1). Thus one obtains, after approximating  $\mathcal{H}_k$  by  $H_k^W$ ,

$$\langle q'' | U(-i\tau_1, -i\tau_2) | q' \rangle \approx \int \prod_{k=1}^N \frac{dq_k dp_k}{2\pi\hbar} \frac{dp_0}{2\pi\hbar} \exp\left\{ \frac{1}{\hbar} \sum_{k=1}^{N+1} [ip_{k-1}(q_k - q_{k-1}) - \frac{\delta}{2}(H_k^W(q_k, p_{k-1}) + H_k^W(q_{k-1}, p_{k-1}^*))] \right\}. \quad (6.4)$$

The Wick rotated version of the quantum action (6.3) is of course

$$\frac{1}{\hbar} \int_{\tau_2}^{\tau_1} d\tau [ip(\tau)\dot{q}(\tau) - \frac{1}{2}(H^W(q_>(\tau), p(\tau), \tau) + H^{W*}(q_<(\tau), p(\tau), \tau))]. \quad (6.5)$$

Here we have identified  $q(t) \equiv q(\tau)$  and the complex conjugation in  $H^{W*}$  ignores the  $i$ 's coming from the Wick rotation.

Matrix Elements of Time Ordered Products. For  $T > t_1 > \dots > t_n > -T$

$$\langle q_f | U(T, t_1) q_1 U(t_1, t_2) q_2 \dots q_n U(t_n, -T) | q_i \rangle = \langle q_f | U(T, -T) T[q_1(t_1) q_2(t_2) \dots q_n(t_n)] | q_i \rangle$$

where the Heisenberg picture operators  $q_k(t_k) \equiv U^{-1}(t_k, -T) q_k U(t_k, -T)$  are defined so that Heisenberg and Schrödinger pictures agree at  $t = -T$ . Working with the l.h.s. of this relation we can insert a complete set of coordinate basis states between each pair of  $U$ 's and then replace each operator  $q_k$  by its eigenvalue and each matrix element of  $U$  by its path integral representation. The integrals over the basis labels  $q'_k$  then simply extend the sum over piecewise histories  $q_i \rightarrow q_n \dots \rightarrow q_1 \rightarrow q_f$  to the sum over all histories  $q_i \rightarrow q_f$ . Thus we arrive at

$$\begin{aligned} & \langle q_f | U(T, -T) T[q_1(t_1) q_2(t_2) \dots q_n(t_n)] | q_i \rangle \\ &= \int \mathcal{D}q \mathcal{D}p q(\tau_1) \dots q(\tau_n) \exp\left\{ \frac{1}{\hbar} \int_{-\mathcal{T}}^{\mathcal{T}} d\tau [ip(\tau)\dot{q}(\tau) - H^Q(q(\tau), p(\tau), \tau)] \right\}. \end{aligned}$$

where we have used the Wick rotated version with  $iT = \mathcal{T}$ . Written in this way with the time ordering symbol on the l.h.s. this formula is valid for any time ordering.

In field theory we are really interested in the vacuum (ground state) expectation value of time ordered products. These can be obtained by taking the limit  $\mathcal{T} \rightarrow \infty$ . Then, inserting energy eigenstates at the left

and right, all states but the ground states are exponentially suppressed so we have

$$\begin{aligned} & \langle q_f | U(\infty, -\infty) T[q_1(t_1) q_2(t_2) \cdots q_n(t_n)] | q_i \rangle \\ &= \langle q_f | 0 \rangle \langle 0 | U(\infty, -\infty) T[q_1(t_1) q_2(t_2) \cdots q_n(t_n)] | 0 \rangle \langle 0 | q_i \rangle \\ &= \langle q_f | 0 \rangle \langle 0 | q_i \rangle \langle out | T[q_1(t_1) q_2(t_2) \cdots q_n(t_n)] | in \rangle. \end{aligned}$$

The wave functions that multiply the desired result can be obtained from

$$\langle q_f | U_{E_{xt=0}}(\infty, -\infty) | q_i \rangle = e^{-2\infty E_G} \langle q_f | 0 \rangle \langle 0 | q_i \rangle,$$

which of course has its own path integral representation. Thus by division we obtain, defining energy so that  $E_G = 0$ ,

$$\begin{aligned} & \langle out | T[q_1(t_1) q_2(t_2) \cdots q_n(t_n)] | in \rangle \\ &= \frac{\int \mathcal{D}q \mathcal{D}p \, q(\tau_1) \cdots q(\tau_n) \exp\left\{\frac{1}{\hbar} \int_{-\infty}^{\infty} d\tau [i p(\tau) \dot{q}(\tau) - H^Q(q(\tau), p(\tau), \tau)]\right\}}{\int \mathcal{D}q \mathcal{D}p \, \exp\left\{\frac{1}{\hbar} \int_{-\infty}^{\infty} d\tau [i p(\tau) \dot{q}(\tau) - H_{E_{xt=0}}^Q(q(\tau), p(\tau))]\right\}}. \end{aligned}$$

Coordinate Space Path Integral. A strong motivation for using the path integral formulation for quantum field theory is that it makes possible a more symmetrical treatment of space and time. This symmetry is even more striking after the Wick rotation when the Lorentz group  $O(3,1)$  becomes simply  $O(4)$  the group of rotations in four dimensions. To achieve the full force of this benefit though we would like to be able to use the configuration space action  $\int dt L(q, \dot{q}, t)$  rather than the phase space one. We can do this provided it is possible to “integrate out” the conjugate momenta  $p$ . This is generally possible in quantum field theory because field theoretic Hamiltonians typically only depend on the conjugate momenta  $\Pi$  through an additive term  $\frac{1}{2} \int d^3x \Pi^2$ . Thus the integral over the  $\Pi$  is gaussian and can be explicitly performed. In the language of quantum mechanics, the field theoretic Hamiltonian is always of the form  $p^2/2 + V(q) + [f(q)p + pf(q)]/2 = p^2/2 + V(q) + f(q)p - i\hbar f'(q)/2$ . In this case we can integrate out the  $p$ 's even before the continuum limit which converts ordinary integrals to path integrals:

$$\begin{aligned} & \int \frac{dp_{k-1}}{\sqrt{2\pi\hbar}} \exp\left\{\frac{1}{\hbar} [i p_{k-1}(q_k - q_{k-1}) \right. \\ & \quad \left. - \delta \left( \frac{1}{2} p_{k-1}^2 + \frac{1}{2} (V(q_k) + V(q_{k-1}) + p_{k-1}(f(q_k) + f(q_{k-1}) - \frac{i\hbar}{4}(f'(q_k) - f'(q_{k-1}))) \right) \right\}. \\ &= \frac{1}{\delta^{1/2}} \exp\left\{-\frac{1}{\hbar} \left[ \frac{(q_k - q_{k-1} + \frac{i\delta}{2}(f(q_k) + f(q_{k-1})))^2}{2\delta} \right. \right. \\ & \quad \left. \left. + \frac{\delta}{2} (V(q_k) + V(q_{k-1}) - \frac{i\hbar}{2}(f'(q_k) - f'(q_{k-1}))) \right] \right\}. \end{aligned}$$

So that the path integral expression (6.4) becomes

$$\begin{aligned} \langle q'' | U | q' \rangle &\approx \left( \frac{1}{2\pi\hbar\delta} \right)^{\frac{(N+1)}{2}} \int \prod_{k=1}^N dq_k \exp\left\{-\frac{1}{\hbar} \sum_{k=1}^{N+1} \left[ \frac{(q_k - q_{k-1} + \frac{i\delta}{2}(f(q_k) + f(q_{k-1})))^2}{2\delta} \right. \right. \\ & \quad \left. \left. + \frac{\delta}{2} (V(q_k) + V(q_{k-1}) - \frac{i\hbar}{2}(f'(q_k) - f'(q_{k-1}))) \right] \right\} \\ &\rightarrow \int \mathcal{D}q \exp\left\{-\frac{1}{\hbar} \int_{\tau_2}^{\tau_1} d\tau \left[ \frac{1}{2} \left( \frac{dq}{d\tau} + \frac{i}{2}(f(q_k) + f(q_{k-1})) \right)^2 + V(q(\tau)) \right] \right\}. \end{aligned}$$

Note that the (divergent) prefactor is necessary to obtain the same result for the evolution amplitude as with the usual operator formalism, and it naturally appears when we start the path history formulation in phase

space. However, notice also that the dependence on the evolution time  $(N + 1)\delta$  is exactly of the form that would come from adding a constant  $-(1/2\delta)\ln(2\pi\hbar\delta)$  to the overall energy of the system. Thus since only energy differences are measurable, the physics is insensitive to the presence of this factor. In quantum field theory we could lump this contribution into the zero point vacuum energy we are supposed to subtract in any case. We recognize the coefficient of  $1/\hbar$  in the exponent as the (imaginary time) classical action for the system. It is this configuration space path integral that gives quantum field theory its neatest expression. Then the classical action has the form  $\int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$  with  $\mathcal{L}$  a Lorentz scalar field. The configuration variables of quantum field theory are the fields  $\phi(\mathbf{x}, t)$  one for each point in space time. To define the sum over histories of fields one therefore needs a lattice in space-time. In the Wick-rotated version this could be taken, for example, to be a hypercubic lattice in 4 dimensional Euclidean space. The path integral defined via such a lattice can be taken as the *definition* of quantum field theory. We shall find that perturbation theory can be developed directly from the continuum path integrals, essentially because one can avoid the actual evaluation of the integrals by various tricks.

Gaussian Integrals When evaluating ground state averages of physical quantities in perturbation theory, one can manage to avoid ever having to do a functional integral. This is because the free field functional integral will cancel between numerator and denominator after extracting the source dependence. However there are cases where one needs to know the numerator (or denominator) separately, for example, when one uses the path integral representation of the partition function. Since the free field integral is simply gaussian, we can in fact calculate it.

We start by noting that the general multivariable gaussian ordinary integral is given by

$$\int_{-\infty}^{\infty} \prod_{i=1}^N \left( \frac{dx_i}{\sqrt{\pi}} \right) e^{-\frac{1}{2} \sum_{km} x_k M_{km} x_m} = \prod_i m_i^{-1/2} = \det^{-1/2} M,$$

where  $m_i$  are the eigenvalues of the real symmetric matrix  $M$ . If we always define Euclidean functional integrals in terms of a lattice, this result can be directly applied. Then after taking the continuum limit, we can write for the neutral scalar field

$$\int \mathcal{D}\phi \exp \left\{ - \int d^4x \left[ \frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2 \right] \right\} \equiv \det^{-1/2} [m^2 - \partial^2].$$

A charged scalar field can be decomposed  $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$  in terms of two real scalar fields so the corresponding formula is

$$\int \mathcal{D}\phi \mathcal{D}\phi^* \exp \left\{ - \int d^4x_E [(\partial\phi^\dagger \partial\phi) + m^2 \phi^\dagger \phi] \right\} \equiv \det^{-1} [m^2 - \partial^2].$$

We have already encountered determinants of differential operators in our study of external field problems, for example, the *outin* matrix element for a charged scalar field in the presence of an external gauge field is proportional to  $\det^{-1} [m^2 - (\partial - iQA)^2]$ . Thus we can immediately write the path history representation for this matrix element:

$$\langle out|in \rangle_A = \frac{\int \mathcal{D}\phi \mathcal{D}\phi^* \exp \left\{ - \int d^4x_E [(\partial + iQA)\phi^* (\partial - iQA)\phi + m^2 \phi^* \phi] \right\}}{\int \mathcal{D}\phi \mathcal{D}\phi^* \exp \left\{ - \int d^4x_E [\partial\phi^* \cdot \partial\phi + m^2 \phi^* \phi] \right\}}.$$

Since this is what must be inserted into the gauge field path integral to couple gauge fields to charged fields, this completes the process for converting to path integration language the qft of scalar fields interacting with

gauge fields. To get *outin* matrix elements of time ordered products of fields we use the generating functional

$$\begin{aligned}
& \langle \text{out} | T [ e^{i \int d^4 x (J^* \phi + \phi^\dagger J)} ] | \text{in} \rangle \\
&= \frac{\int \mathcal{D}\phi \mathcal{D}\phi^* \exp \left\{ - \int d^4 x_E [ (\partial + iQA)\phi^* (\partial - iQA)\phi + m^2 \phi^* \phi - J^* \phi - \phi^* J ] \right\}}{\int \mathcal{D}\phi \mathcal{D}\phi^* \exp \left\{ - \int d^4 x_E [ \partial\phi^* \cdot \partial\phi + m^2 \phi^* \phi ] \right\}} \\
&= \frac{\det^{-1}(m^2 - D^2)}{\det^{-1}(m^2 - \partial^2)} e^{\int d^4 x d^4 y i J^*(x) \Delta_F(x, y; A) i J(y)}.
\end{aligned}$$

We have written the functional integrals in Euclidean space, which we indicate with the  $E$  subscript. To express the results in terms of Minkowski space, just use  $d^4 x_E = i d^4 x$ . Remember that  $\Delta_F(x, y; A)$  is precisely the continuation back to Minkowski space of the Euclidean Green function  $\Delta_E(x_E, y_E; A)$ . To confirm the last equality one simply completes the square and changes variables in the by now familiar way. Doing this in Euclidean space leads to the exponent  $\int d^4 x_E d^4 y_E J^*(x) \Delta_E(x_E, y_E; A) J(y)$ , which, when continued back to Minkowski space, gives the result shown.

We know from our experience with the operator formulation that the boundary conditions that fix  $\Delta_F$  in Minkowski space are the requirements that it contains only positive frequencies at very late times and only negative frequencies at very early times. In Euclidean space, the boundary condition on  $\Delta_E$  is simply that it vanish at infinity. Since  $\Delta_F, \Delta_E$  are analytic continuations of one another, either statement of the boundary conditions is satisfactory. From the path integral point of view it is more natural to choose the Euclidean version.

Anticommuting Fields To convert the persistence amplitude for the Dirac field in the presence of external gauge fields to a path integral formalism, we must be able to produce a factor of  $\det\{m - i\gamma \cdot D\}$  in the *numerator* unlike the denominator as gaussian integrals tend to produce. This requires the introduction of “anti-commuting” numbers or  $a$  numbers in contrast to ordinary  $c$  numbers. Any two  $a$  numbers  $e, f$  satisfy  $ef + fe = 0$ . In particular the square of an  $a$  number vanishes! Thus the most general function of a single  $a$  number  $a$  is the linear one  $c_1 + c_2 a$ . The theory of functions of  $a$  numbers is quite trivial. A function of  $N$   $a$  numbers is at most linear in each variable, but that involves terms with up to  $N$  factors. With  $N \rightarrow \infty$  one can of course have any number of factors as long as each factor is a different  $a$  number.

How do we integrate over  $a$  numbers? To define this we define  $\int da f(a)$  to be a linear operation that assigns a unique  $c$  number to each function  $f$ . We also require the fundamental translation property  $\int da f(a + e) = \int da f(a)$ . But there are only two linearly independent functions of  $a$  namely 1 and  $a$  itself. So we only need to specify  $\int da 1$  and  $\int da a$ . The translation property for the second of these holds only if  $\int da 1 = 0$ . Thus we only need to specify  $\int da a$  to be some fixed  $c$  number, and then integration is completely defined! By definition we take  $\int da a = 1$ . Then the integral of the arbitrary function  $c_1 + c_2 a$  is simply  $c_2$ , the coefficient of the linear term in  $a$ .

Now consider integration of a gaussian over  $2M$   $a$  numbers,  $a_k, \bar{a}_k$ :

$$\begin{aligned}
& \int da_1 d\bar{a}_1 \cdots da_M d\bar{a}_M e^{\bar{a}^T C a} \\
&= \frac{1}{M!} \int da_1 d\bar{a}_1 \cdots \int da_M d\bar{a}_M (\bar{a}^T C a)^M \\
&= \int da_1 d\bar{a}_1 \cdots \int da_M d\bar{a}_M \bar{a}_1 C_{1k_1} a_{k_1} \bar{a}_2 C_{2k_2} a_{k_2} \cdots \bar{a}_M C_{1k_M} a_{k_M} \\
&= C_{1k_1} C_{2k_2} \cdots C_{1k_M} \epsilon^{k_1 k_2 \cdots k_M} = \det C.
\end{aligned}$$

This shows that integrating gaussians over anticommuting numbers yields determinants with positive powers.

Next consider the case where the exponent of the gaussian is a bilinear in  $M$   $a$  numbers  $b_k: \frac{1}{2}b^T A b$  where  $A$  is an antisymmetric matrix. It can be reduced to the previous case by considering its square:

$$\begin{aligned} & \left( \int db_1 \dots db_M e^{b^T A b / 2} \right)^2 = \int db_1 \dots db_M db'_1 \dots db'_M e^{(b^T A b + b'^T A b') / 2} \\ & = (-)^{\frac{M^2}{2}} \int da_1 d\bar{a}_1 \dots da_M d\bar{a}_M e^{\bar{a}^T A a} \\ & = (-)^{\frac{M^2}{2}} \det A \end{aligned}$$

where the change of variables  $a = (b + ib')/\sqrt{2}$ ,  $\bar{a} = (b - ib')/\sqrt{2}$  has been used. The phase out front is never relevant, since when  $M$  is odd, both sides vanish. We conclude that

$$\int db_1 \dots db_M e^{b^T A b / 2} = \det^{1/2} A.$$

Since the l.h.s. is a polynomial in the matrix elements of  $A$  we have proved an interesting corollary that the determinant of an antisymmetric matrix is the square of a polynomial in the matrix elements. That polynomial is sometimes known as the Pfaffian,  $Pf(A) = \det^{1/2}[A]$  for antisymmetric matrices  $A$ .

Now we can repeat our discussion of bosonic functional integrals for the fermionic case:

$$\begin{aligned} \langle out | in \rangle_A &= \frac{\det(m - i\gamma \cdot D)}{\det(m - i\gamma \cdot \partial)} \\ &= \frac{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left\{ -i \int d^4x (\bar{\psi}(m - i\gamma \cdot D)\psi) \right\}}{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left\{ -i \int d^4x (\bar{\psi}(m - i\gamma \cdot \partial)\psi) \right\}}. \end{aligned}$$

Introducing anticommuting sources,  $\eta, \bar{\eta}$  in the combination  $i \int d^4x [\bar{\eta}\psi + \bar{\psi}\eta]$  in the exponent and competing the square, one can easily see that the source dependence is just a factor

$$e^{i \int \bar{\eta}(m - i\gamma \cdot D)^{-1} \eta} = e^{\int d^4x d^4y i \bar{\eta}(x) S_F(x, y; A) \eta(y)}.$$

By differentiating with respect to the sources, one can show that this expression is just the generating function for the outin matrix element of time ordered products of fields:

$$\begin{aligned} & \langle out | T e^{i \int d^4x [\bar{\eta}\psi + \bar{\psi}\eta]} | in \rangle \\ &= \frac{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left\{ -i \int d^4x [(\bar{\psi}(m - i\gamma \cdot D - \bar{\eta}\psi - \bar{\psi}\eta)\psi)] \right\}}{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left\{ -i \int d^4x (\bar{\psi}(m - i\gamma \cdot \partial)\psi) \right\}} \\ &= \frac{\det(m - i\gamma \cdot D)}{\det(m - i\gamma \cdot \partial)} e^{\int d^4x d^4y i \bar{\eta}(x) S_F(x, y; A) \eta(y)}. \end{aligned}$$

Gauge Fields The path integral for gauge theories presents special problems because of the gauge invariance of the action. We shall establish the proper formulation for Quantum Electrodynamics by first obtaining the path integral in Coulomb gauge  $\nabla \cdot \mathbf{A} = 0$ , following the general procedure sketched in this chapter. Once that has been done, we can discuss within the new path integral formalism more general gauges, including covariant ones.

The Hamiltonian operator for charged fields interacting with the quantized e.m. field  $\mathbf{A}$  in Coulomb gauge is (as we shall see in the next chapter)

$$H_{eff} = \int d^3x \left( \frac{1}{2} \mathbf{\Pi}_T^2 + \frac{1}{2} \partial_k \mathbf{A}_T \cdot \partial_k \mathbf{A}_T - \mathbf{A}_T \cdot \mathbf{J}_e \right) + \int (J_e^0 + j^0) \left( -\frac{1}{2\nabla^2} \right) (J_e^0 + j^0) + H_{fields}(\mathbf{A})|_{A_0=0}, \quad (6.6)$$

Here we understand all operators to be in Schrödinger picture. We see that this is a case where the conjugate momentum appears only quadratically so we can immediately transcribe an infinitesimal evolution between eigenstates of  $\mathbf{A}$  near imaginary time  $\tau$

$$\begin{aligned} & \langle \mathbf{A}'' | e^{-d\tau H_{eff}} | \mathbf{A}' \rangle \\ & \approx \delta(\nabla \cdot \mathbf{A}'') \exp \left\{ d\tau \int d^3x \left[ \frac{1}{2} (\dot{\mathbf{A}})^2(\tau) + \frac{1}{2} (\partial \mathbf{A})^2(\tau) - \mathbf{A}(\tau) \cdot \mathbf{J}_e \right] \right. \\ & \quad \left. + d\tau \int (J_e^0 + j^0) \left( -\frac{1}{2\nabla^2} \right) (J_e^0 + j^0) + d\tau H_{fields}(\mathbf{A}(\tau))|_{A_0=0} \right\}, \end{aligned}$$

where the terms involving  $\mathbf{A}(\tau)$  with no time derivative are averaged over  $\mathbf{A}(\tau) = \mathbf{A}''$ ,  $\mathbf{A}'$  and  $\dot{\mathbf{A}}(\tau)$  symbolizes the continuum limit of  $(\mathbf{A}'' - \mathbf{A}')/d\tau$ . The delta function multiplying the r.h.s. symbolizes the condition  $\nabla \cdot \mathbf{A} = 0$ . It is necessary if we wish to formally integrate over all three components of  $\mathbf{A}$ . Notice that for the moment we are only changing the e.m. field into path integral language, leaving the charged fields as operators.

Next we employ a widely used trick for simplifying the term describing the instantaneous Coulomb interaction. This involves introducing an auxiliary variable  $A^4(\mathbf{x}, t)$  and writing the identity

$$e^{-d\tau \int (J_e^0 + j^0) \left( -\frac{1}{2\nabla^2} \right) (J_e^0 + j^0)} = \frac{\int \mathcal{D}A^4(\tau) \exp \left\{ -d\tau \int d^3x \left[ \frac{1}{2} (\nabla A^4)^2 - A^4 (J_e^0 + j^0) \right] \right\}}{\int \mathcal{D}A^4(\tau) \exp \left\{ -d\tau \int d^3x \frac{1}{2} (\nabla A^4)^2 \right\}}$$

where  $j^4 = ij^0$  and  $J_e^4 = iJ_e^0$  as appropriate after the Wick rotation. This identity is proved by completing the square by a shift of integration variable  $A^4 \rightarrow A^4 - (1/\nabla^2)j^4$ . Here we use the fact that ultimately  $d\tau \rightarrow 0$  to neglect any commutators that might arise due to the fact that  $j^0$  is really an operator. (This latter approximation is in exactly the same spirit as the replacement of  $\mathcal{H}_k$  by  $H_k^W$  in our general derivation of the path integral.) The variable  $A^4$  enters now exactly as the imaginary time component of the vector potential would have entered before it was eliminated by solving the Gauss' law constraint. For example, using  $\nabla \cdot \mathbf{A} = 0$  we can write

$$\int d^3x \left[ \frac{1}{2} \dot{\mathbf{A}}^2 + \frac{1}{2} (\nabla \mathbf{A}_k)^2 - \mathbf{A} \cdot \mathbf{J}_e \right] + \frac{1}{2} (\nabla A^4)^2 - A^4 J_e^0 = \int d^3x \left[ \frac{1}{4} F_{\mu\nu} F_{\mu\nu} - A_\mu J_e^\mu \right].$$

Furthermore  $H_{fields}|_{A^4=0} - \int d^3x A^4(\tau) j^4 = H_{fields}$  with  $A^4 \neq 0$  playing the role of the imaginary time component of the vector potential.

Composing many infinitesimal evolutions, we arrive at the path integral representation

$$\langle \mathbf{A}_f | U(\infty, -\infty) | \mathbf{A}_i \rangle = \frac{\int \mathcal{D}A_\mu \delta(\nabla \cdot \mathbf{A}) e^{-\int d^4x (F_{\mu\nu} F_{\mu\nu}/4 - A_\mu J_e^\mu)} \langle f | T e^{-\int d\tau H_{fields}(A_\mu(\tau))} | i \rangle}{\int \mathcal{D}A^4 e^{-\int d\tau (\nabla A^4)^2/2}}.$$

As discussed before, by taking the evolution over an infinite time interval we effectively project onto the ground states (vacuum) at early and late times. We can therefore pick convenient initial and final states, *e.g.* we can let them be eigenstates of  $\mathbf{A}$  with vanishing eigenvalue and ground states of  $H_{fields}(A_\mu(\tau = \mp\infty), \mp\infty)$  respectively. We then obtain the following general formula for the *outin* matrix element of time ordered products of fields

$$\langle out | T[\psi(y_1) \cdots \psi(y_m) \bar{\psi}(z_m) \cdots \bar{\psi}(z_1)] | in \rangle_{J_e, A_e} = \frac{\int \mathcal{D}A_\mu \delta(\nabla \cdot \mathbf{A}) e^{-\int d^4x (F_{\mu\nu} F_{\mu\nu}/4 - A_\mu J_e^\mu)} \langle out | T[\psi(y_1) \cdots \bar{\psi}(z_1)] | in \rangle_{A(\tau) + A_e(\tau)}}{\int \mathcal{D}A_\mu \delta(\nabla \cdot \mathbf{A}) e^{-\int d^4x F_{\mu\nu} F_{\mu\nu}/4} \langle out | in \rangle_{A(\tau)}}.$$

An important observation is that the set of operators in the time ordered product on the l.h.s. can be expanded to include any number of vector potentials by functionally differentiating w.r.t. the external current  $J_e^\mu$ . This is clear from the pure exponential dependence on  $J_e$  on the r.h.s. Thus in addition to describing the presence of real external sources, the  $J_e$  dependence provides a generating function for all correlation functions of any number of vector potentials in the source free case: simply set  $J_e = 0$  *after* differentiating the appropriate number of times w.r.t.  $J_e$ . To illustrate this point, consider the path integral for the free e.m. field (no charged fields).

$$\langle out | in \rangle_{J_e} = \frac{\int \mathcal{D}A_\mu \delta(\nabla \cdot \mathbf{A}) e^{-\int d^4x (F_{\mu\nu} F_{\mu\nu}/4 - A_\mu J_e^\mu)}}{\int \mathcal{D}A_\mu \delta(\nabla \cdot \mathbf{A}) e^{-\int d^4x F_{\mu\nu} F_{\mu\nu}/4}}.$$

We can extract the dependence on  $J_e$  by shifting the integration variable in the numerator by  $A_\mu \rightarrow A_\mu + C_\mu$ , where  $C$  is chosen so that the linear term in  $A$  is cancelled. It must of course also be restricted by the Coulomb gauge condition  $\nabla \cdot \mathbf{C} = 0$ . After the shift the coefficient of the linear term in  $A$  becomes after an integration by parts

$$\partial_\mu (\partial_\mu C_\nu - \partial_\nu C_\mu) + J_\nu = 0.$$

This equation is only consistent if the external current is conserved  $\partial_\mu J^\mu = 0$  which we are assuming. For  $\nu = 4$ , remembering the Coulomb gauge condition, this equation determines  $C^4 = (-1/\nabla^2) J^4$ . After using current conservation to write  $\partial_4 C^4 = (1/\nabla^2) \nabla \cdot \mathbf{J}_e$  the spatial components are determined to be  $\mathbf{C} = (-1/\partial^2)(\mathbf{J} - \nabla(1/\nabla^2) \nabla \cdot \mathbf{J})$ . The question of boundary conditions is settled in Euclidean space by requiring that  $C$  vanish in all four directions at infinity. As we have discussed this prescription becomes the familiar  $i\epsilon$  one when continued back to Minkowski space. Inserting these results into the path integral we find

$$\begin{aligned} \langle out | in \rangle_{J_e} &= e^{-\int d^4x (F_{\mu\nu}^C F_{\mu\nu}^C/4 - C_\mu J_e^\mu)} \\ &= e^{\int d^4x C_\mu J_e^\mu/2} \\ &= e^{\frac{1}{2} \int d^4x d^4y J_e^\mu(x) D_{\mu\nu}(x-y) J_e^\nu(y)}. \end{aligned}$$

where

$$D_{jk}(x) = \int \frac{d^4 p}{(2\pi)^4} e^{ix \cdot p} \frac{\delta_{jk} - p^j p^k / \mathbf{p}^2}{p^2}$$

$$D_{44}(x) = \int \frac{d^4 p}{(2\pi)^4} e^{ix \cdot p} \frac{1}{\mathbf{p}^2} = \delta(t) \frac{1}{4\pi|\mathbf{x}|}$$

$$D_{4k} = D_{k4} = 0.$$

To get correlation functions of any number of vector potentials we differentiate  $\langle out|in \rangle_{J_e}$  with respect to the current any number of times. Because of current conservation we may only differentiate w.r.t. three components of  $J^\mu$  say the two transverse components  $\mathbf{J}_T$  and  $J^4$ . But that is sufficient since the longitudinal component of  $\mathbf{A}$  is zero by the gauge condition. If we set  $J_e = 0$  after differentiating, it is easy to check that the Wick expansion follows with the two point function

$$\langle A_\mu(x) A_\nu(y) \rangle = D_{\mu\nu}(x - y).$$

Note that with path integrals it is more precise to speak of correlation functions, which are functional averages of some number of fields, than of the vacuum expectations of time ordered products of field operators. They are of course numerically equal to each other.

General Gauges The path integral formulation of gauge theories is particularly suited to the discussion of general gauges. To keep the formalism covariant we would like to be able to replace the noncovariant Coulomb gauge condition with a covariant one such as  $\partial_\mu A_\mu = 0$ . The procedure for gauge fixing that follows is due to Fadeev and Popov, and is quite general. The method starts by selecting some general gauge fixing condition  $F(A, \partial A) = 0$ . This condition should have the property that for any value of the gauge potential it is possible to find a gauge transformation to a potential for which  $F = 0$ , and further that if  $F(A) = 0$  then  $F(A_\Omega) \neq 0$  with  $\Omega$  any nontrivial gauge transformation which vanishes at infinity. We don't require this property for more general gauge transformations because that would rule out Coulomb gauge which seems to be perfectly adequate. The path integrand in such a gauge should contain a factor of  $\delta(F(A))$ .

The F-P procedure is to define a functional  $\Delta_F(A)$  by the requirement

$$1 = \Delta_F(A) \int \mathcal{D}\Omega \delta(F(A_\Omega))$$

where  $A_\Omega$  is the transformation of  $A$  by the gauge group element  $\Omega(x)$ , and the measure  $\mathcal{D}\Omega$  is gauge invariant. By this definition  $\Delta_F$  is clearly gauge invariant. Now insert this representation for 1 in the "unfixed" gauge field path integrand. Next change functional integration variables so that  $A_\Omega \rightarrow A$ . Here we implicitly assume that the unfixed measure  $\mathcal{D}A_\mu$  is invariant under changes of variables which are gauge transformations<sup>\*</sup>. Then the infinite volume of the gauge group  $\int \mathcal{D}\Omega$  comes out as a common factor in both the numerator and denominator of the functional average and so cancels. We are left with the factors

$$\Delta_F(A) \delta(F(A))$$

in the functional integrand. The delta function fixes the gauge and the factor  $\Delta_A$  is in general needed to guarantee that different choices for  $F$  yield the same answer for gauge invariant quantities. (It is only for functional averages of gauge invariant quantities that the rest of the integrand stays invariant under the variable change that removes  $\Omega$  from the delta function.)

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\* In a completely general context this assumption might clash with the more basic translational invariance of the measure. In such a case there would be a Jacobian accompanying the variable change. This is not needed for the usual abelian and nonabelian gauge theories however.

Our Coulomb gauge path integral did not include a factor of  $\Delta_{Coul}(A)$  so we need to confirm that it is inconsequential for the abelian case. For this we have to compute

$$\frac{1}{\Delta_{Coul}} = \int \mathcal{D}\Lambda \delta(\nabla \cdot \mathbf{A} + \nabla^2 \Lambda) = \frac{1}{\det(-\nabla^2)}.$$

We see that  $\Delta_{Coul}$  is independent of  $A$  so that it will cancel between the numerator and denominator of functional averages. Thus our failure to put it in gives no differences in physical quantities. Thus all gauge choices are formally equivalent to Coulomb gauge provided the Fadeev-Popov factor  $\Delta_F(A)$  is included along with the gauge fixing delta functional.

In practice  $\Delta_F(A)$  always multiplies  $\delta(F(A))$  so the former is only needed for  $A$  satisfying the gauge condition. For such  $A$ ,  $\delta(F(A_\Omega))$  only contributes for infinitesimal  $\Omega$ :

$$F(A(x) + \delta A(x)) \approx \int d^4 y \frac{\delta F(A(x))}{\delta A_{\mu ab}(y)} \delta A_{\mu ab}(y).$$

Denoting the infinitesimal generators of the gauge group by  $G(z)$ ,  $\delta A$  is linear in the matrix elements of  $G$ :  $\delta A_{\mu ab}(y) = \int d^4 z \mathcal{L}_{\mu ab, cd}(y, z; A) G_{cd}(z)$  where  $\mathcal{L}$  is a linear differential operator depending on  $A$  in general. For example, the infinitesimal nonabelian gauge transformation reads  $\delta A = \partial G - ig[A, G]$ . Thus  $\Delta_F = \det(\frac{\delta F}{\delta A_\mu} \mathcal{L}_\mu)$  where the determinant is that of a linear differential operator which is also a matrix in the internal group space with matrix elements labeled  $(ab, ef)$ :

$$\int d^4 y \frac{\delta F_{ab}(A(x))}{\delta A_{\mu cd}(y)} \mathcal{L}_{\mu cd, ef}(y, z; A).$$

An important class of covariant gauges consists of the Lorentz gauges  $\partial \cdot A = f$  where  $f$  is some fixed function. The F-P determinant for this gauge is  $\det(-\partial^2)$  in the abelian case of QED. Notice that it is independent of both  $f$  and  $A$ . In the nonabelian case, it would depend upon  $A$  of course, but is still independent of  $f$ . So for QED the F-P determinant cancels between numerator and denominator. and it is safe to ignore it<sup>†</sup>. Functional averages of gauge invariant quantities will be independent of  $f$ . We can exploit this to get rid of the functional delta function in the path integrand by averaging over  $f$  with a gaussian weight function  $e^{-\int d^4 x f^2(x)/2\alpha}$ . This just multiplies the numerator and denominator by the same constant so it won't alter gauge invariant quantities. The net effect of this is to remove the delta function and instead add a non gauge invariant term  $\int d^4 x (\partial \cdot A)^2(x)/2\alpha$  to the Euclidean action. For  $\alpha = 1$  (Feynman Gauge) the effect of this term is to simplify the kinetic term for  $A$ :

$$\int d^4 x \left( \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2\alpha} (\partial \cdot A)^2 \right) = \int d^4 x \left( \frac{1}{2} \partial_\mu A_\nu \partial_\mu A_\nu + \frac{1-\alpha}{2\alpha} (\partial \cdot A)^2 \right)$$

$$\xrightarrow{\alpha \rightarrow 1} \int d^4 x \frac{1}{2} \partial_\mu A_\nu \partial_\mu A_\nu$$

Just as with the Coulomb gauge, the photon propagator for the “ $\alpha$ ” gauges is obtained by evaluating the path integral for the gauge field in the presence of an external source. The source dependence is easily

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<sup>†</sup> There are some applications where it is nonetheless important to keep it. For example, it contributes a constant to the zero point energy which just subtracts the spurious contribution of the time-like and longitudinal components of the vector potential. Also when one computes the finite temperature partition function in a covariant gauge, the F-P factor removes two photon degrees of freedom so the total number is  $4 - 2 = 2$ . Without the F-P factor the Stefan-Boltzmann law would be off by a factor of 2!

obtained by shifting  $A \rightarrow A + C$  with  $C$  satisfying

$$\partial^2 C_\mu + \frac{1-\alpha}{\alpha} \partial \partial \cdot C = -J^\mu$$

which is solved first for  $\partial \cdot C = -\alpha \cdot J$  and then

$$C_\mu = -\frac{1}{\partial^2} (J_\mu - (1-\alpha) \frac{\partial_\mu \partial_\nu}{\partial^2} J^\nu).$$

Thus the propagator is

$$D_{\mu\nu}^\alpha(x) = \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot x} \frac{\delta_{\mu\nu} - (1-\alpha) p_\mu p_\nu / p^2}{p^2}.$$

In the continuation to Minkowski space  $\delta_{\mu\nu}$  is replaced by  $\eta_{\mu\nu}$ ,  $p^2$  by  $p^2 - i\epsilon$ , and there is an additional factor of  $-i$ . We note the great simplification for Feynman gauge. The case  $\alpha = 0$ , known as Landau gauge is effectively the gauge  $\partial \cdot A = 0$  because the coefficient of the gauge breaking term in the action blows up damping out all contributions to the integral not satisfying this condition. We see that in this case the divergence of the propagator vanishes,  $\partial_\mu D_{\mu\nu}^0 = 0$ . The fundamental reason that QED is independent of  $\alpha$  is that  $A$  always couples to a conserved current so that the terms involving  $\alpha$  in the propagator decouple.

To get a bit more insight into the role of current conservation consider a photon propagator attached to two conserved vertices in momentum space

$$Amp \sim \frac{R_\mu^1 R^{2\mu}}{p^2} = \frac{\mathbf{R}^1 \cdot \mathbf{R}^2 - R_0^1 R_0^2}{p^2}.$$

It would seem that this coupling corresponds to the propagation of four photon states, 3 “space” components and 1 time component, the last one coupling with the “wrong” sign. These wrong sign states are sometimes called “ghosts.” But by current conservation  $R_0^a = \mathbf{p} \cdot \mathbf{R}^a / p^0$ . In the limit of physical photon momentum, *i.e.* for which  $p^2 = 0$  the residue of the pole is just

$$\mathbf{R}^1 \cdot \mathbf{R}^2 - \frac{\mathbf{p} \cdot \mathbf{R}^1 \mathbf{p} \cdot \mathbf{R}^2}{p^2}$$

which is to say only the transverse states (perpendicular to  $\mathbf{p}$ ) truly propagate.

## 7. The Dirac Equation

In our development of relativistic quantum mechanics we shall study intensively the case of spin 1/2 particles solved by Dirac. After this we shall return to the problem of higher spin. As a practical matter, the Dirac equation contains a tremendous amount of the present understanding of elementary particles. In fact, all of the known fundamental constituents of matter are either spin 1/2 (quarks and leptons), or spin 1 gauge particles (photon, gluons, weak vector bosons). To these we may have to add a fundamental scalar (Higgs particle) and of course there is always the spin 2 graviton that must be understood eventually. But we can go a tremendous distance to understanding the physics of the standard model through study of the Dirac equation interacting with a gauge field. Indeed, for many applications the gauge field doesn't even need to be quantized!

Dirac's original motivation for his equation is still useful and inspiring to recall. The most straightforward attempt to write down a relativistic version of the Schrödinger equation

$$\begin{aligned} i\hbar \frac{\partial \psi(\mathbf{x}, t)}{\partial t} &= \sqrt{m^2 c^4 - (\hbar c)^2 \nabla^2} \psi(\mathbf{x}, t) \\ &\equiv m \sum_{k=0}^{\infty} \binom{1/2}{k} \left( \frac{-\hbar^2 \nabla^2}{m^2} \right)^k \psi(\mathbf{x}, t) \end{aligned} \quad (7.1)$$

is horribly nonlocal, involving an arbitrarily high number of spatial derivatives. Dirac proposed to get around this by making  $\psi$  a multicomponent wave function and defining the square root in a local way by using matrices, in analogy with the properties of the Pauli spin matrices

$$(\boldsymbol{\sigma} \cdot \mathbf{v})^2 = \mathbf{v}^2.$$

To achieve this he required the introduction of four anticommuting matrices  $\gamma^\mu$  with  $\mu = 0, 1, 2, 3$ :

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}. \quad (7.2)$$

Using (7.2) it is simple to show that

$$(\gamma^\mu \partial_\mu)^2 = \frac{1}{2} \{\gamma^\mu, \gamma^\nu\} \partial_\mu \partial_\nu = -\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2,$$

where we have chosen units for which  $c = 1$ . We shall also choose units so that  $\hbar = 1$ . Then the Dirac equation is

$$\frac{1}{i} \gamma^\mu \partial_\mu \psi + m\psi = 0. \quad (7.3)$$

To cast this equation as a relativistic Schrödinger equation we rewrite it as

$$i \frac{\partial \psi(\mathbf{x}, t)}{\partial t} = \left( \frac{1}{i} \boldsymbol{\alpha} \cdot \nabla + \beta m \right) \psi(\mathbf{x}, t)$$

where we have multiplied through by  $\gamma^0 \equiv \beta$  using  $\beta^2 = I$  and have defined  $\boldsymbol{\alpha} \equiv \gamma^0 \boldsymbol{\gamma}$ .

So far we have not specified the gamma matrices. It is simple to show that they must be at least  $4 \times 4$  (In  $D$  space-time dimensions the minimum size is  $2^{D/2} \times 2^{D/2}$  for even  $D$ .)

Let  $\sigma^k$ ,  $k = 1, 2, 3$  be the  $2 \times 2$  Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we have two popular representations:

**Standard Representation of  $\gamma^\mu$**

**Chiral (Natural) Representation of  $\gamma^\mu$**

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} & \gamma^0 &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \\ \boldsymbol{\gamma} &= \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} & \boldsymbol{\gamma} &= \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \\ \gamma_5 &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} & \gamma_5 &= \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \\ \sigma_{km} &= \epsilon_{kmn} \begin{pmatrix} \sigma^n & 0 \\ 0 & \sigma^n \end{pmatrix} & \sigma_{km} &= \epsilon_{kmn} \begin{pmatrix} \sigma^n & 0 \\ 0 & \sigma^n \end{pmatrix} \end{aligned}$$

where  $\epsilon_{kmn}$  is the completely antisymmetric three tensor with  $\epsilon_{123} = +1$ . The standard representation is more convenient for slowly moving particles, whereas the chiral one is more convenient for massless fermions that move at the speed of light.

Clearly  $\beta = \gamma^0$  is hermitian,  $\boldsymbol{\gamma}$  is anti-hermitian, and  $\boldsymbol{\alpha}$  is hermitian. Thus the Hamiltonian

$$H = \frac{1}{i} \boldsymbol{\alpha} \cdot \nabla + \beta m \quad (7.4)$$

is a hermitian operator as it should be.

To justify the interpretation of the Dirac particle as spin 1/2, we need to construct an angular momentum operator for which  $\boldsymbol{\alpha}$  transforms as a vector. This is clearly accomplished by choosing the spin operator to be  $\boldsymbol{\Sigma}/2$  with

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

Then the Dirac Hamiltonian obviously commutes with the total angular momentum

$$\mathbf{J} = \mathbf{r} \times \frac{1}{i} \nabla + \frac{\boldsymbol{\Sigma}}{2}.$$

The Dirac wave function transforms under the  $\frac{1}{2} \oplus \frac{1}{2}$  representation of the rotation group. In particular it describes a spin 1/2 particle.

To get the energy spectrum of the Dirac particle we must find all the eigenstates of the Hamiltonian

(7.4). Clearly, it is best to work with momentum eigenstates

$$\psi_{\mathbf{p}} = u(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}},$$

since the momentum operator commutes with  $H$ . Then the coefficient spinor must satisfy

$$(\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)u(\mathbf{p}) = Eu(\mathbf{p}). \quad (7.5)$$

Since the square of the matrix on the l.h.s. is just  $m^2 + \mathbf{p}^2$ , we immediately learn that

$$E(\mathbf{p}) = \pm\sqrt{m^2 + \mathbf{p}^2}.$$

It is also fairly simple to find the eigenfunctions in the Standard Representation. It is sufficient to find the positive energy eigenfunctions, because the negative energy ones can be obtained from them by a simple operation: Complex conjugating (7.5) and multiplying both sides by  $i\gamma^2$ , we learn that  $i\gamma^2 u^*(-\mathbf{p})$  is an eigenstate of  $H$  with eigenvalue  $-E$  if  $u(\mathbf{p})$  is an eigenstate with eigenvalue  $+E$ . An explicit solution for  $u$ , with  $E = +\omega(\mathbf{p}) > 0$ , is easily shown to be

$$u(\mathbf{p}) = \begin{pmatrix} \phi \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{m+\omega(\mathbf{p})}\phi \end{pmatrix} \quad (7.6)$$

where  $\omega = \sqrt{m^2 + \mathbf{p}^2}$  and  $\phi$  is any two component spinor. The two independent components just represent the two spin states of a spin 1/2 particle. As we have explained the negative energy eigenfunctions are then

$$u_-(\mathbf{p}) = i\gamma^2 u^*(-\mathbf{p}). \quad (7.7)$$

The positive energy states of the Dirac particle give the desired relativistic description of a relativistic spin 1/2 particle. The negative energy states are a disaster for the single particle interpretation of the Dirac wave function. As soon as the Dirac particle in a positive energy state is subjected to external forces transitions to negative energy states will be induced and the system is unstable. Dirac himself proposed the remedy for this problem, which exploits the Pauli Exclusion Principle for fermions: If one postulates that the Dirac particle is a fermion, then one can consider the “vacuum” to be the state in which *all* of the negative energy levels are filled, then any further particles added to this state must by the exclusion principle occupy only positive energy states! This vacuum state is sometimes called the “negative energy sea.” Although we call this state the vacuum, it is clear from its construction that it is far from empty, since it contains an infinite number of particles. The presence of the sea will make itself felt as soon as one considers interactions of the Dirac particles with themselves or with independent force fields. Strictly speaking, there is an enormous (infinitely) negative energy associated with the sea. However in quantum mechanics, only energy differences are measurable (in the absence of gravity) and we might as well measure the energy of all states relative to that of the sea, *i.e.* we take  $E_{sea} = 0$ . The total momentum contained in the sea is automatically zero because momentum states in all directions are occupied.

In addition if the Dirac particle is charged the sea possesses an infinite uniform positive charge density. It is convenient in this case to postulate a compensating negative background charge density, so our vacuum will be neutral

To understand the dynamical significance of the presence of the sea, it is instructive to consider the excitation spectrum of the whole system. We can first of all add  $N$  particles with momenta  $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N$  to the sea. The resulting state will have energy

$$E_{N \text{ particles}} = \sum_k \sqrt{m^2 + \mathbf{p}_k^2}$$

and  $N$  units of charge.

Next we can *remove*  $N$  particles with momenta  $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N$  from the sea. This state, which we describe as a state of  $N$  holes, will also have a positive energy

$$E_{N \text{ holes}} = \sum_k \sqrt{m^2 + \mathbf{q}_k^2}$$

since we have subtracted negative energy. However the momentum of this state is

$$\mathbf{P}_{N \text{ holes}} = - \sum_k \mathbf{q}_k$$

for the same reason, and the charge is  $-N$  units. Thus this state appears to be a state of  $N$  particles of opposite charge to the ordinary (positive energy) Dirac particle and with momenta  $-\mathbf{q}_1, -\mathbf{q}_2, \dots, -\mathbf{q}_N$ . This is how antiparticles appear in the theory.

Finally, one can simply excite  $N$  particles from the sea to positive energy states. This excitation does not change the number of particles so it is neutral. The energy of this state is the sum of the positive energy eigenvalues occupied minus the sum of the negative energy levels vacated. It is a state with  $N$  particles and  $N$  antiparticles (holes), with corresponding energy. In principle, this state can be prepared from the sea by delivering sufficient energy, at least  $2Nm$ , to the system.

If the original Dirac particle is the electron with charge  $-e$ , then the theory predicts the existence of its antiparticle, the positron, with charge  $+e$ . The positron was unknown when Dirac discovered his equation, and the idea of identifying the positron with the proton failed because the latter does not have the electron's mass nor could it annihilate with an electron into photons. Thus when the positron was eventually discovered, it was a spectacular confirmation of Dirac's theory.

## 8. Second Quantization

We have seen that the consistent interpretation of the Dirac theory requires the presence of the sea of an infinite number of negative energy particles. In practice, however, at least if interactions are weak, all but a finite number of particles in the sea are spectators in any calculation. Thus we need a formalism that allows us to concentrate only on the part of the sea that is active in a given process. The formalism which achieves this has been developed, and for historical reasons goes under the name of second quantization. It should be stressed that it is completely equivalent to a description in terms of the many body Schrödinger wave function for the sea, but it is much less cumbersome, and almost indispensable to efficient calculation. The formalism is applicable to any system of identical particles, bosons or fermions. We shall present only the fermion case in lecture since we will only be applying it to the Dirac theory. The changes needed for the boson case will be indicated in a series of exercises.

We begin with the concept of the occupation number basis. Consider a system of fermions, and let an orthonormal complete set of states for a single fermion be labeled by an index  $\alpha$ :

$$\psi_\alpha(\mathbf{x}).$$

Then a multi-fermion antisymmetrized tensor product state is completely specified by the set of numbers  $\{n_\alpha\}$  which give the number of fermions (0 or 1) occupying each state  $\alpha$ .

Now the crucial idea of second quantization is to define a creation operator  $b_\alpha^\dagger$  by

$$b_\alpha^\dagger |n_1 \dots n_\alpha \dots\rangle = (-)^{\sum_{\gamma < \alpha} n_\gamma} |n_1 \dots n_\alpha + 1 \dots\rangle.$$

The state dependence of the prefactor is necessary for consistency with the antisymmetry of the state under fermion interchange. Since  $b_\alpha^\dagger$  adds a fermion in state  $\alpha$  its square must vanish. It is a direct consequence of this fact and its definition that

$$\{b_\alpha^\dagger, b_\beta^\dagger\} = 0$$

By considering

$$\langle \{n\} | b_\alpha^\dagger | \{n'\} \rangle^* = \langle \{n'\} | b_\alpha | \{n\} \rangle$$

it is clear that  $b_\alpha$  removes a fermion from the state  $\alpha$  and so is an annihilation operator. It is then simple to check that

$$\{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha\beta}.$$

For our application, the state label will include a continuous momentum as well as a discrete label. In that case the Kronecker delta will of course include a factor of a Dirac delta function  $\delta(\mathbf{p} - \mathbf{p}')$ .

Having introduced the creation and annihilation operators, we now have a very efficient notation for the occupation number basis. Call the state with no states occupied  $|0'\rangle$ . Then the state with

$$n_{\alpha_1} = n_{\alpha_2} = \dots n_{\alpha_N} = 1$$

and all other occupation numbers zero is just

$$b_{\alpha_N}^\dagger \dots b_{\alpha_2}^\dagger b_{\alpha_1}^\dagger |0'\rangle.$$

We are now in a position to relate all this formalism to the standard many-body Schrödinger wave function description. The wave function describing the state in which the single particle states  $\alpha_1 \dots \alpha_N$  are

occupied is just

$$\Psi_{\alpha_1 \dots \alpha_N}(\mathbf{x}_1, \dots, \mathbf{x}_N) = \frac{1}{\sqrt{N!}} \sum_P \delta_P \psi_{\alpha_1}(\mathbf{x}_{P(1)}) \psi_{\alpha_2}(\mathbf{x}_{P(2)}) \dots \psi_{\alpha_N}(\mathbf{x}_{P(N)})$$

where  $\delta_P = -1$  if  $P$  is an odd permutation and  $+1$  if  $P$  is an even permutation. If we define the field operator

$$\psi(\mathbf{x}) \equiv \sum_{\alpha} b_{\alpha} \psi_{\alpha}(\mathbf{x}), \quad (8.1)$$

with anticommutation relations

$$\begin{aligned} \{\psi(\mathbf{x}), \psi(\mathbf{y})\} &= \{\psi^{\dagger}(\mathbf{x}), \psi^{\dagger}(\mathbf{y})\} = 0 \\ \{\psi(\mathbf{x}), \psi^{\dagger}(\mathbf{y})\} &= \sum_{\alpha} \psi_{\alpha}(\mathbf{x}) \psi_{\alpha}^*(\mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), \end{aligned}$$

Then we have

$$\Psi_{\alpha_1 \dots \alpha_N}(\mathbf{x}_1, \dots, \mathbf{x}_N) = \frac{1}{\sqrt{N!}} \langle 0' | \psi(\mathbf{x}_1) \psi(\mathbf{x}_2) \dots \psi(\mathbf{x}_N) b_{\alpha_N}^{\dagger} \dots b_{\alpha_2}^{\dagger} b_{\alpha_1}^{\dagger} | 0' \rangle.$$

A completely general  $N$  body wave function is of course a general superposition of such states

$$\Phi^N = \sum_{\alpha_1 \dots \alpha_N} c_{\alpha_1 \dots \alpha_N} \Psi_{\alpha_1 \dots \alpha_N} = \frac{1}{\sqrt{N!}} \langle 0' | \psi(\mathbf{x}_1) \psi(\mathbf{x}_2) \dots \psi(\mathbf{x}_N) | \Phi \rangle$$

where

$$|\Phi\rangle = \sum_{\alpha_1 \dots \alpha_N} c_{\alpha_1 \dots \alpha_N} b_{\alpha_N}^{\dagger} \dots b_{\alpha_2}^{\dagger} b_{\alpha_1}^{\dagger} | 0' \rangle$$

The state  $|\Phi\rangle$  thus can describe the most general many body Schrödinger wave function. Indeed, the second quantization formalism allows one to even superpose states of different numbers of particles, so it is the more general description!

We have shown how to describe general quantum states, it remains to show how linear operators are related between the two formalisms. All possible observables in the Schrödinger wave function description are completely symmetric in the operators acting on each particle. Such operators can be classified according to the number of particles involved in each term of the operator. For example, a one body operator acting on  $N$  identical particles has the form

$$\Omega^{(1)} = \sum_k \omega_k$$

where  $\omega_k$  acts only on the variables of the  $k^{\text{th}}$  particle. A two body operator would be a sum of terms, each acting on a pair of particles:

$$\Omega^{(2)} = \sum_{k < m} \omega_{km}.$$

A  $K$  body operator is obviously a sum of terms each acting on  $K$  particles. In a typical nonrelativistic system with  $N$  identical particles, *e.g.* the electrons of a  $Z = N$  atom, the kinetic energy of the electrons is a 1 body operator, while the potential energy is a two body operator (since the electrons interact in pairs). The interaction of the electrons with an external field would be described by another one body operator.

The second quantized description of these operators is straightforward. Starting with the one body case we first represent the single particle operator  $\omega$  by its matrix elements in the single particle basis we have introduced

$$\omega_{\alpha\beta} = \langle \alpha | \omega | \beta \rangle.$$

Then the one body operator  $\Omega^{(1)}$  is just

$$\Omega^{(1)} = \sum_{\alpha\beta} b_{\alpha}^{\dagger} \omega_{\alpha\beta} b_{\beta}.$$

To see that this definition has the correct action on multi-particle states, one simply applies it to one of the basis states

$$\Omega^{(1)} b_{\alpha_N}^{\dagger} \cdots b_{\alpha_2}^{\dagger} b_{\alpha_1}^{\dagger} |0'\rangle = \sum_k b_{\alpha_N}^{\dagger} \cdots \sum_{\gamma_k} (b_{\gamma_k}^{\dagger} \langle \gamma_k | \omega | \alpha_k \rangle) \cdots b_{\alpha_1}^{\dagger} |0'\rangle$$

and we see that the r.h.s. is a sum of terms for the  $k^{th}$  of which  $\omega$  has the correct action on the  $k^{th}$  single particle state label.

Multi-body operators have the obvious analogous second quantized description. We limit the discussion here to the two-body case. Let  $|\beta_1\rangle |\beta_2\rangle$  denote the standard unsymmetrized two particle tensor product state. It is convenient to reverse the order of factors in the corresponding bra  $\langle \alpha_2 | \langle \alpha_1 |$  so that

$$(\langle \alpha_2 | \langle \alpha_1 | | \beta_1 \rangle | \beta_2 \rangle) = \delta_{\alpha_1 \beta_1} \delta_{\alpha_2 \beta_2}.$$

Then we define

$$\omega_{\alpha_2 \alpha_1, \beta_1 \beta_2} = \langle \alpha_2 | \langle \alpha_1 | \omega | \beta_1 \rangle | \beta_2 \rangle$$

and the two-body operator  $\Omega^{(2)}$  is then just

$$\Omega^{(2)} = \frac{1}{2} \sum_{\alpha_1 \alpha_2, \beta_1 \beta_2} b_{\alpha_2}^{\dagger} b_{\alpha_1}^{\dagger} \omega_{\alpha_2 \alpha_1, \beta_1 \beta_2} b_{\beta_1} b_{\beta_2}.$$

A typical example which requires both one and two body operators is the hamiltonian for a system of  $N$  nonrelativistic fermions with an interaction potential energy  $V(\mathbf{r}_k, \mathbf{r}_m) = V(\mathbf{r}_m, \mathbf{r}_k)$  between each pair  $(k, m)$ . The Schrödinger picture Hamiltonian for this system is just

$$H = \sum_k \frac{-\hbar^2}{2m} \nabla_k^2 + \sum_{k < m} V(\mathbf{r}_k, \mathbf{r}_m).$$

According to the procedure just outlined, the second quantized version of the kinetic term is

$$\sum_{\alpha\beta} b_{\alpha}^{\dagger} \langle \alpha | \left( \frac{-\hbar^2}{2m} \nabla_k^2 \right) | \beta \rangle b_{\beta} = \int d^3 r \psi^{\dagger}(\mathbf{r}) \left( \frac{-\hbar^2}{2m} \nabla^2 \right) \psi(\mathbf{r}) = \int d^3 r \frac{\hbar^2}{2m} \nabla \psi^{\dagger}(\mathbf{r}) \cdot \nabla \psi(\mathbf{r}),$$

where in the second form we have gone to the coordinate basis, using the definition of the second quantized field operator (8.1). The second quantized version of the potential term is the two-body operator

$$\frac{1}{2} \sum_{\alpha_1 \alpha_2, \beta_1 \beta_2} b_{\alpha_2}^{\dagger} b_{\alpha_1}^{\dagger} \langle \alpha_2 | \langle \alpha_1 | V | \beta_1 \rangle | \beta_2 \rangle b_{\beta_1} b_{\beta_2} = \frac{1}{2} \int d^3 x d^3 y V(\mathbf{x}, \mathbf{y}) \psi^{\dagger}(\mathbf{y}) \psi^{\dagger}(\mathbf{x}) \psi(\mathbf{x}) \psi(\mathbf{y}).$$

Two details to note about this expression are the overall factor of 1/2 and the order of operators. These are necessary to arrange that the potential energy has the correct sign and normalization. The complete

hamiltonian for this system can now be compactly written as

$$H = \int d^3r \frac{\hbar^2}{2m} \nabla \psi^\dagger(\mathbf{r}) \cdot \nabla \psi(\mathbf{r}) + \frac{1}{2} \int d^3x d^3y V(\mathbf{x}, \mathbf{y}) \psi^\dagger(\mathbf{y}) \psi^\dagger(\mathbf{x}) \psi(\mathbf{x}) \psi(\mathbf{y}).$$

This is the hamiltonian for a quantum field theory. The fundamental quantum operators are the local fields  $\psi(\mathbf{x})$  or the corresponding creation and annihilation operators. The operators  $x, p$  etc. of the Schrodinger description have been demoted to  $c$  number labels and derivatives with respect to them.

One final feature of the formalism to explain is the role of the field equations. In the Schrödinger picture the quantum dynamics is given by the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Phi, t\rangle = H_S(t) |\Phi, t\rangle$$

where we stress that we allow time varying external forces to be present (hence the time dependence of  $H$ ). The field equations arise in the Heisenberg picture wherein the time dependence resides in the operators rather than in the system states which are constant in time. To pass to the Heisenberg picture we write

$$|\Phi, t\rangle = U(t) |\Phi, 0\rangle$$

where

$$i\hbar \frac{\partial}{\partial t} U = H_S(t) U \quad U(0) = I,$$

and give the time independent Schrödinger picture operators  $\Omega$  time dependence according to

$$\Omega(t) \equiv U^\dagger \Omega U.$$

The Heisenberg picture Hamiltonian is similarly related to the Schrödinger picture one by

$$H(t) \equiv U^\dagger H_S(t) U.$$

Then the Heisenberg picture operators corresponding to constant Schrödinger picture operators satisfy the Heisenberg equations

$$i\hbar \dot{\Omega}(t) = [\Omega(t), H(t)].$$

Returning to our system of nonrelativistic fermions we find that the Heisenberg equation for the field operator  $\psi$  implies

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = \frac{-\hbar^2}{2m} \nabla^2 \psi(\mathbf{x}, t) + \int d^3z V(\mathbf{x}, \mathbf{z}) \psi^\dagger(\mathbf{z}) \psi(\mathbf{z}) \psi(\mathbf{x}). \quad (8.2)$$

This is the quantum field equation. It is a nonlinear differential equation for a quantum operator, and because of its operator nature it has much more information packed in it than is immediately apparent. The origin of the name “second quantization” for this formalism is that (8.2) looks like a nonlinear version of the Schrödinger equation. Indeed if the particles did not mutually interact, it would be exactly the Schrödinger equation. If “first quantization” produced the Schrödinger equation, we have now reinterpreted the latter as a classical field equation, which is then “second quantized”. Of course we know that all we have really done is given a clever reformulation of ordinary many body quantum mechanics, and the name is really a misnomer, which has stuck.

We should mention that (8.2) is an elegant point of departure for the Hartree-Fock approximation, in which one approximates the nonlinear term by a one body term

$$\int d^3y K(\mathbf{x}, \mathbf{y}) \psi(\mathbf{y}) \quad (8.3)$$

in which  $K$  is chosen to make this term as close as possible to the original two body term.

How much of the quantum dynamics is captured in the quantum field equation? The time dependent many body wave function in Schrödinger picture, which summarizes the complete quantum dynamics of the system can be recast

$$\frac{1}{\sqrt{N!}} \langle 0' | \psi(\mathbf{x}_1) \cdots \psi(\mathbf{x}_N) | \Phi, t \rangle = \frac{1}{\sqrt{N!}} \langle 0' | U(t) \psi(\mathbf{x}_1, t) \cdots \psi(\mathbf{x}_N, t) | \Phi, 0 \rangle.$$

So in addition to the time dependence of  $\psi(\mathbf{x}, t)$  which we could get from the field equation, we would also need to find the time dependence of the state

$$\langle 0' | U(t).$$

In our example, the state  $\langle 0' |$  is really dynamically inert *i.e.*

$$\langle 0' | U(t) = \langle 0' |,$$

because each term of  $H_S(t)$  has a  $b^\dagger$  on the left so that  $\langle 0' | H_S(t) = 0$ . Thus knowledge of  $\psi(\mathbf{x}, t)$  at all times allows us to reconstruct the time dependence of the many body Schrödinger wave function completely. If we were transforming a different matrix element, in which  $\langle 0' |$  were replaced by a nonempty state, to the Heisenberg picture, the time dependence of  $U$  acting on this state would have to be found in addition to the time dependence of the Heisenberg operators. The time dependence of  $\langle A | U(t)$  has a simple interpretation if  $\langle A |$  is characterized as an eigenstate, say, of some definite Schrödinger picture observable  $\Omega$ . Then  $\langle A | U(t)$  is the corresponding eigenstate of the corresponding Heisenberg picture operator  $\Omega(t) = U^\dagger \Omega U$ . In particular if the time dependent terms in the Schrödinger picture Hamiltonian  $H_S(t) = H_0 + H'(t)$  vanish initially and finally,

$$H'(T) = H'(0) = 0,$$

and  $|A\rangle$  is an eigenstate of  $H_0 = H_S(0) = H(0)$ , then  $\langle A | U(T)$  is an eigenstate of the Heisenberg picture operator  $H_0(T) = H(T)$ .

Let us consider what is gained and lost in the alternative formulations of many body quantum physics. The second quantization formalism contains all the information contained in the (anti)symmetrized wave functions. The (unphysical) non-symmetrized wave functions are of course lost, but that is desirable. The second quantization machinery allows a broader range of dynamical options. For example, particle number conservation is built into the wave function description. This conservation law is reflected in the existence of the number operator

$$N = \sum_{\alpha} b_{\alpha}^{\dagger} b_{\alpha} = \int d^3x \psi^{\dagger}(\mathbf{x}) \psi(\mathbf{x}) \quad (8.4)$$

which counts the number of particles in a given state. The conservation law is the statement that  $N$  commutes with the Hamiltonian:  $[N, H] = 0$ . We can also identify  $\psi^{\dagger}(\mathbf{x}) \psi(\mathbf{x})$  as the number density operator. There is, however, no principle which excludes considering a Hamiltonian with terms that don't commute with  $N$ . For example, a term

$$\int d^3x d^3x' A(\mathbf{x}, \mathbf{x}') (\psi(\mathbf{x}) \psi(\mathbf{x}') - \psi^{\dagger}(\mathbf{x}) \psi^{\dagger}(\mathbf{x}'))$$

would not commute with  $N$ .