

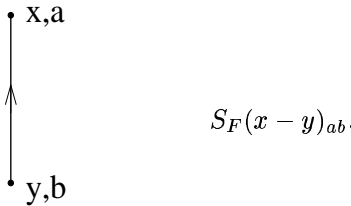
14. External Field Perturbations

As an important example, let us apply perturbation theory to the Dirac field in an external electromagnetic field. Since we shall work in interaction picture throughout, we shall not append the I subscript to indicate interaction picture: that will be understood throughout this section. Then the persistence amplitude in the presence of A is

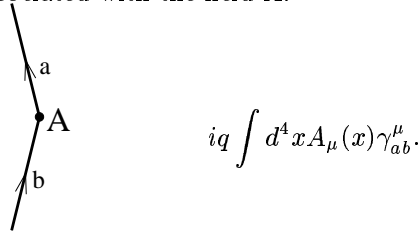
$$\begin{aligned} \langle out|in \rangle_A &= \langle 0| T e^{i \int d^4x j^\mu(x) A_\mu(x)} |0\rangle \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\frac{g}{2}\right)^n \int d^4x_1 \cdots d^4x_n A_{\mu_1} \cdots A_{\mu_n} \\ &\quad \langle 0| T [[\bar{\psi}(x_1), \gamma^{\mu_1} \psi(x_1)] \cdots [\bar{\psi}(x_n), \gamma^{\mu_n} \psi(x_n)]] |0\rangle. \end{aligned}$$

It is now a matter of applying Wick's theorem to evaluate the n^{th} term of the series.

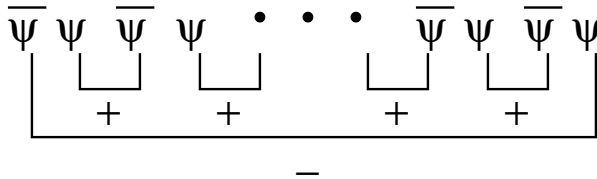
We shall organize the calculation with Feynman diagrams. The Wick expansion expresses the time ordered product in terms of propagators completely characterized by two points. Each propagator is represented by a line connecting the two points, directed from the argument of $\bar{\psi}$ to that of ψ :



The lines terminate on vertices associated with the field A :



For each closed loop there is a trace over Dirac indices and a multiplicative factor of -1 . This last factor is due to the anticommuting property of the Dirac field and arises because the order of fields in the product of currents that is contracted to form the loop differs from that of the contributing propagators by an odd number of interchanges.



Finally there is a combinatoric factor arising from a sometimes partial cancellation of the $1/n!$ multiplying the n^{th} order term.

Connected Diagrams Terms in the perturbation series described by disconnected diagrams factorize into a product of the values of each connected subdiagram. Thus all the useful information is contained in the subset of connected diagrams, and it is useful to know how the final answer is expressed in terms of connected diagrams only. Roughly speaking, the sum of all diagrams is simply the exponential of the sum

of all connected diagrams. This statement applies to the expansion of

$$\langle 0| T e^{-i \int dt H_I(t)} |0\rangle \quad (14.1)$$

for a completely general Hamiltonian. The reason is purely combinatoric. Call the value of the sum of connected graphs at order k , $G_k^c/k!$. Consider the terms at order $n = \sum_{k=1}^{\infty} k r_k$ in the perturbation series described by r_k appearances of the connected graphs of order k . (Note that all but a finite number of the r_k are 0.) The contractions leading to these terms can occur in $n!/\prod_k [(k!)^{r_k} r_k!]$ ways. This is because there are $n!$ ways to assign the n Hamiltonians to the n vertices of all the connected subgraphs, but this overcounts the number of contractions by a factor of $k!$ for each of the connected subgraphs of order k , because different orders of the assignment to each subgraph do not give distinct contractions, and overcounts by a factor of $r_k!$ for each group of identical subgraphs for the same reason. Thus the value of the order n terms is

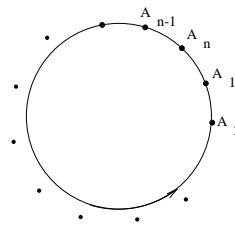
$$\frac{1}{n!} \sum_{\sum k r_k = n} \frac{n!}{\prod_k [(k!)^{r_k} r_k!]} \prod_k (G_k^c)^{r_k} = \sum_{\sum k r_k = n} \prod_k \left[\frac{1}{r_k!} \left(\frac{G_k^c}{k!} \right)^{r_k} \right].$$

Note that the factors in square brackets are 1 if all $r_k = 0$. Summing over all n simply relaxes the constraint on the summation over the r_k , so we have finally

$$\langle 0| T e^{-i \int dt H_I(t)} |0\rangle = \prod_{k=1}^{\infty} \left[\sum_{r_k=0}^{\infty} \frac{1}{r_k!} \left(\frac{G_k^c}{k!} \right)^{r_k} \right] = \prod_{k=1}^{\infty} e^{G_k^c/k!} = e^{\sum_k G_k^c/k!},$$

which is the desired result.

We now turn to the connected diagrams for an external electromagnetic field. At order n in the fields, the diagrams contributing to $\ln \langle out|in \rangle$ are



$$= -(iq)^n \int d^4 x_1 \cdots d^4 x_n \text{Tr}[\gamma \cdot A(x_1) S_F(x_1 - x_2) \cdots \gamma \cdot A(x_n) S_F(x_n - x_1)]. \quad (14.2)$$

Since there are $(n-1)!$ distinct diagrams with the same value (after the coordinate integrations) the net combinatoric factor is $(n-1)!/n! = 1/n$. This factor can be interpreted as $1/S_\Gamma$ where S_Γ is the symmetry number of the graph γ . It is also worth noting that the same formula applies to the case of nonabelian gauge field $A_{\mu ab}$ with the understanding that the trace includes the trace over the internal indices as well as the spinor indices.

In fact, it is instructive to regard the coordinates as (continuous) indices, so $S_{F_{x_1 a, x_2 b}}$ is a matrix and $(\gamma \cdot A)_{x_1 x_2} \equiv \delta(x_1 - x_2) \gamma \cdot A$ is a matrix, so the term as a whole can be regarded as a grand trace

$$-\frac{1}{n} \text{Tr}[iq \gamma \cdot A S_F]^n$$

and the sum over all n is then recognized as the Taylor expansion for a logarithm:

$$\ln \langle out|in \rangle_A = \text{Tr}[\ln(I - iq \gamma \cdot A S_F)].$$

Making use of the identities $\det A = \exp\{\text{Tr} \ln A\}$ and $\det AB = \det A \det B$, which are fundamental prop-

erties of the determinant, and noting that in this matrix notation $S_F = -i(m + (1/i)\gamma \cdot \partial)^{-1}$, we see that

$$\begin{aligned} \langle out|in \rangle_A &= \det(I - q\gamma \cdot A(m + \frac{1}{i}\gamma \cdot \partial)^{-1}) \\ &= \frac{\det(m + \frac{1}{i}\gamma \cdot \partial - q\gamma \cdot A)}{\det(m + \frac{1}{i}\gamma \cdot \partial)} \\ &\equiv \frac{\det(m - i\gamma \cdot D)}{\det(m - i\gamma \cdot \partial)} \end{aligned} \quad (14.3)$$

where we have defined the covariant derivative operator as $D = \partial - iqA$. The denominator in (14.3) serves to normalize $\langle out|in \rangle$ to 1 at $A = 0$, which amounts to choosing the zero of energy to be the ground state energy of H_0 . In practice these “explicit formulae” for $\langle out|in \rangle$ can not be evaluated exactly for general A , although for special external potentials such as those corresponding to constant field strength it is possible. Nonetheless they give the most efficient derivation of the perturbation series in powers of the external fields, and as we shall see give some general insight into the meaning of the amplitudes we are calculating.

Furry’s Theorem. There appears to be a connected diagram (14.2) for every n . But for the electromagnetic case, only those with even n are nonvanishing. To see why this is true use the trace property $\text{Tr} A^T = \text{Tr} A$ to show that

$$\begin{aligned} \text{Tr}[\gamma \cdot A(x_1)S_F(x_1 - x_2) \cdots \gamma \cdot A(x_n)S_F(x_n - x_1)] = \\ \text{Tr}[S_F^T(x_n - x_1)\gamma^T \cdot A(x_n) \cdots S_F^T(x_1 - x_2)\gamma^T \cdot A(x_n)]. \end{aligned}$$

But $\gamma^{\mu T} = -(\gamma^0\gamma^2)^{-1}\gamma^\mu\gamma^0\gamma^2$. From this it follows that

$$S_F^T(x) = -i(\gamma^0\gamma^2)^{-1} \int \frac{d^4p}{(2\pi)^4} \frac{m + \gamma \cdot p}{m^2 + p^2} e^{ip \cdot x} \gamma^0\gamma^2 = (\gamma^0\gamma^2)^{-1} S_F(-x)\gamma^0\gamma^2$$

So we have

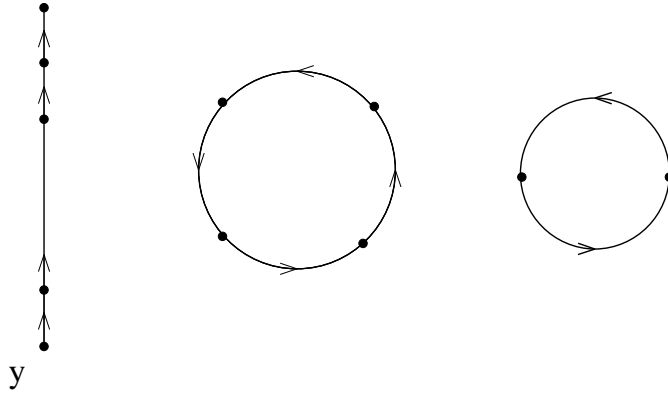
$$\begin{aligned} \text{Tr}[\gamma \cdot A(x_1)S_F(x_1 - x_2) \cdots \gamma \cdot A(x_n)S_F(x_n - x_1)] = \\ (-)^n \text{Tr}[S_F(x_1 - x_n)\gamma \cdot A(x_n) \cdots S_F(x_2 - x_1)\gamma \cdot A(x_1)]. \end{aligned}$$

After integrating over the x ’s the only difference between the left and right sides is the labeling of dummy integration variables and the factor $(-)^n$ on the right. Thus for odd n both sides must vanish, *i.e.* all connected diagrams for $\langle out|in \rangle$ with an odd number of A ’s vanish. This is Furry’s theorem. A more basic way to understand the result is to note that the substitution $A \rightarrow -A$ can be undone by the charge conjugation transformation under which $j \rightarrow -j$. Thus $\langle out|in \rangle$ must be an even function of A .

The perturbation series can be similarly “summed” for the propagator in the presence of external gauge fields

$$\begin{aligned} \langle out|T\psi(x)\bar{\psi}(y)|in \rangle_A = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n \\ \langle 0|T[A \cdot j(x_1) \cdots A \cdot j(x_n)\psi(x)\bar{\psi}(y)]|0 \rangle. \end{aligned}$$

The connected subdiagrams are of two types: (1) In a given term there are any number of closed loop diagrams of the sort contributing to $\langle out|in \rangle_A$; (2) In each term there is exactly one subdiagram with a continuous line running from y to x



Summing up all of the first type of diagram gives simply an overall factor of $\langle out|in \rangle_A$ multiplying the value of each diagram of the second type. Apart from the net combinatoric factor the latter diagram has at order n the value

$$\begin{aligned}
 & \begin{array}{c} x \\ \bullet \\ \uparrow \\ \bullet \\ \uparrow \\ \bullet \\ \uparrow \\ \bullet \\ \uparrow \\ \bullet \\ y \end{array} \\
 & = -(iq)^n \int d^4 x_1 \cdots d^4 x_n \\
 & S_F(x - x_1) \gamma \cdot A(x_1) S_F(x_1 - x_2) \cdots \gamma \cdot A(x_n) S_F(x_n - y).
 \end{aligned} \tag{14.4}$$

In fact the combinatoric factor is just 1 since there are $n!$ distinct contractions leading to this diagram (giving identical values since they only differ in the labeling of dummy integration variables), and that precisely cancels the $1/n!$ coming from the Dyson formula. In the matrix notation introduced above, we recognize the sum of all diagrams, including the disconnected closed loops, as

$$\begin{aligned}
 \langle out|in \rangle_A \sum_{n=0}^{\infty} S_F(iq\gamma \cdot AS_F)^n &= \langle out|in \rangle_A S_F \frac{1}{I - iq\gamma \cdot AS_F} \\
 &= \langle out|in \rangle_A \frac{-i}{m - i\epsilon - i\gamma \cdot D}.
 \end{aligned}$$

In other words the right hand side is just proportional to a Green function for the differential operator $m - i\gamma \cdot D$:

$$\langle out|T\psi(x)\bar{\psi}(y)|in \rangle_A = \langle out|in \rangle_A S(x, y; A)$$

where

$$(m - i\gamma \cdot D)S(x, y; A) = -i\delta^4(x - y),$$

and the boundary condition is specified by giving an infinitesimal negative imaginary part to m . Since $A = 0$ in the distant past and future, this prescription corresponds to the requirement of only positive frequencies as $t \rightarrow +\infty$ and negative frequencies as $t \rightarrow -\infty$, which are manifest properties of $\langle out|T\psi(x)\bar{\psi}(y)|in \rangle_A$. The fact that this latter quantity is this Green function is also immediately seen by applying the differential operator and remembering the contribution from differentiating the step functions implicit in the time ordering symbol.

By making use of our result for $\langle out|T\psi(x)\bar{\psi}(y)|in\rangle_A$, we can give a much quicker derivation of the determinant formula for $\langle out|in\rangle_A$. Looking back to Eq(13.4) we see that if we make a change δA in A , the change in $\langle out|in\rangle$ is

$$\delta \langle out|in\rangle_A = \langle in|\delta U(\infty, -\infty)|in\rangle = i \int d^4x \delta A_\mu(x) \langle out|j^\mu(x)|in\rangle_A.$$

On the other hand the current matrix element can be related to the Green function of the previous paragraph

$$\begin{aligned} \langle out|j^\mu(x)|in\rangle_A &= \langle out|\frac{q}{2}[\bar{\psi}(x), \gamma^\mu \psi(x)]|in\rangle_A \\ &= -\frac{q}{2} \text{Tr} \gamma^\mu \langle out|[\psi(x), \bar{\psi}(x)]|in\rangle_A \\ &= -q \text{Tr} \gamma^\mu \langle out|T[\psi(x)\bar{\psi}(x)]|in\rangle_A \\ &= -q \text{Tr} \gamma^\mu S(x, x; A) \langle out|in\rangle_A. \end{aligned}$$

where we are interpreting $\theta(0) = 1/2$. Using $\delta \langle out|in\rangle / \langle out|in\rangle = \delta \ln \langle out|in\rangle$ we obtain

$$\begin{aligned} \delta \ln \langle out|in\rangle_A &= -iq \int d^4x \text{Tr}[\gamma \cdot \delta A S(x, x; A)] \\ &= \text{Tr}[-q\gamma \cdot \delta A (m - i\gamma \cdot D)^{-1}] \\ &= \delta \text{Tr} \ln[m - i\gamma \cdot D] \end{aligned}$$

which implies our previous formula up to a multiplicative A independent constant which is fixed by requiring $\langle out|in\rangle_{A=0} = 1$. Actually this initial condition is somewhat artificial; it would be more natural to simply take $\langle out|in\rangle_A = \det(m - i\gamma \cdot D)$. This corresponds to not removing the sea contribution to the energy in the case $A = 0$.

It is instructive to see how the formal expression for the sea energy comes out of this evaluation. When $A = 0$ the matrix element (in momentum basis) $\langle p'|\ln(m - \gamma \cdot \partial)|p\rangle = \delta(p' - p) \ln(m + \gamma \cdot p)$, so when we take the trace over the continuous momentum indices, we set $p' = p$ and get an overall factor of $\delta^4(0)$. To interpret this singular factor, think of the integral representation for $\delta(p) = \int e^{ip \cdot x} d^4x / (2\pi)^4$. As $p \rightarrow 0$ this is just $VT/(2\pi)^4$ where V is the volume of space and T the duration of time. With this interpretation, we have

$$\begin{aligned} \text{Tr} \ln(m - i\gamma \cdot \partial) &= \frac{VT}{(2\pi)^4} \int d^4p \text{Tr} \ln(m + \gamma \cdot p) \\ &= \frac{VT}{(2\pi)^4} \int d^4p 2 \ln(m^2 + p^2) \end{aligned}$$

where we have used the fact that the eigenvalues of $\gamma \cdot p$ are $+\sqrt{-p^2}$ twice and $-\sqrt{-p^2}$ twice. To compare this to the sea energy we obtained earlier, we need to interpret the integral over p^0 . This is of course a divergent integral, but if we differentiate once with respect to m^2 to get the mass dependence, we get a convergent integral which can be evaluated by closing the contour in the upper half complex p^0 plane

$$\int \frac{dp^0}{2\pi} \frac{1}{\omega(\mathbf{p})^2 - p^{02} - i\epsilon} = \frac{i}{2\omega} = i \frac{d}{dm^2} \omega(\mathbf{p}).$$

Thus

$$\ln \langle out|in\rangle = 2iT \frac{V}{(2\pi)^3} \int d^3p [\omega(\mathbf{p}) + C]$$

where the constant is at least independent of the mass. Since $\langle out|in\rangle \sim e^{-iE_{sea}T}$ we see that we recover our previous result for E_{sea} at least as far as the mass dependence is concerned. The constant C is itself infinite and complex: the Wick rotation to imaginary p^0 gives a factor of i , but the contours at infinity are nonvanishing and complex.

The occurrence of complex energies in the Minkowski definition of $\langle out|in \rangle$ motivates the idea that these vacuum amplitudes should be defined fundamentally in Euclidean space and then continued back to Minkowski space as the application demands. In Euclidean space-time we put $T = -iT_E$ and the vacuum amplitude would have the behavior $\langle out|in \rangle \sim \exp(-E_{sea}T_E)$. If we repeat the calculation of $\mathbf{Tr} \ln(m - i\gamma \cdot \partial)$ in Euclidean space we get $VT_E \int d^4p_E 2 \ln(m^2 + p^2)/(2\pi)^4$ which is manifestly real albeit divergent, and we get the formula

$$E_{sea} = -2 \frac{V}{(2\pi)^4} \int d^4p_E \ln(m^2 + p^2).$$

This still has the mass dependence of our earlier result and the mass independent discrepancy is at least real. In effect, working in Euclidean space-time from the beginning discards undesirable complex contours from Wick rotations that fail to vanish only because of the poor high momentum (ultra-violet) behavior of the integrands of vacuum diagrams. Consequently, among modern field theorists Euclidean space-time is widely accepted as the best way to define vacuum amplitudes.

Finally, we consider the calculation of time ordered products of an arbitrary number of Dirac fields. Clearly the nonvanishing ones have an equal number of ψ 's and $\bar{\psi}$'s:

$$\langle out|T[\psi(x_1) \cdots \psi(x_n) \bar{\psi}(y_n) \cdots \bar{\psi}(y_1)]|in \rangle.$$

To each term there will correspond any number of connected closed loop diagrams and n connected diagrams of exactly the type contributing to the $n = 2$ case. Each of these latter subdiagrams consists of a line from one of the y_k to one of the x_{P_k} . The closed loops sum to an overall factor of $\langle out|in \rangle$. For each distinct pairing of the x 's with the y 's the other subdiagrams sum to a product of n factors $S(x_{P_k}, y_k; A)$. In other words, there is a Wick expansion for time ordered products of Dirac fields in the presence of an external field exactly as in the $A = 0$ case. All one does is substitute $S_F(x - y) \rightarrow S(x, y; A)$ and multiply by an overall factor of $\langle out|in \rangle_A$.


15. Scattering in External Fields

Our time dependent formalism is ideally suited for defining transition amplitudes. Since external fields are turned off in the far future and distant past, particle states can be defined in exactly the same manner as the *in* and *out* vacua. We can expand the Heisenberg field operators at early(late) times in terms of annihilation and creation operators $b_\lambda^{in}(\mathbf{p}), d_\lambda^{in}(\mathbf{p})$ ($b_\lambda^{out}(\mathbf{p}), d_\lambda^{out}(\mathbf{p})$), since they are free fields there. These are of course eigenoperators for $H_0(-\infty)(H_0(+\infty))$ respectively. They are of course determined up to a phase by the labels λ, \mathbf{p} . We can fix the phases of the *out* operators in terms of the *in* operators by defining $\Omega^{out} \equiv U_I^{-1}(\infty, -\infty)\Omega^{in}U_I(\infty, -\infty)^*$. Then incoming particle states are obtained by applying $b^{in\dagger}, d^{in\dagger}$ to $|in\rangle$ and outgoing particle states are obtained by applying b^{out}, d^{out} to $\langle out|$.

The transition amplitudes between multiparticle states at early times and multiparticle states at late times can be immediately transcribed to interaction picture:

$$\langle out| b_1^{out} \dots d_N^{out} d_M^{in\dagger} \dots b_1^{in\dagger} |in\rangle = \langle 0, I| b_{I1} \dots d_{IN} U_I(\infty, -\infty) d_{IM}^\dagger \dots b_{I1}^\dagger |0, I\rangle.$$

The rules for expanding these amplitudes in perturbation theory are very similar to those for the outin matrix elements of time ordered products of Dirac field operators. The disconnected closed loops sum up to an overall factor of $\langle out|in\rangle_A$. The creation and annihilation operators can either contract against each other or against one of the Dirac fields in U_I :



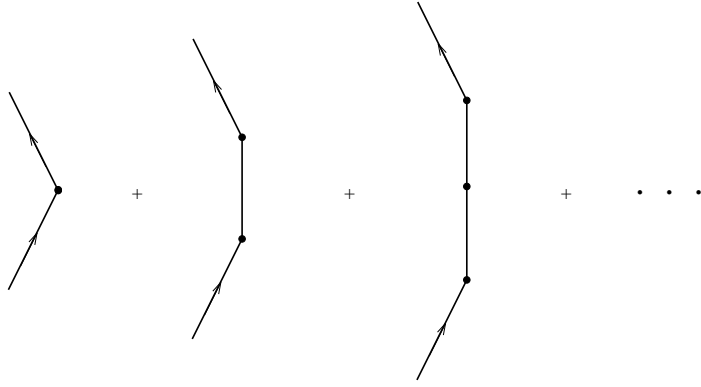
$$\begin{aligned} \langle 0, I| b_\lambda(\mathbf{p}) \bar{\psi}(x) |0, I\rangle &= \frac{1}{(2\pi)^{3/2} \sqrt{2\omega}} \bar{u}_\lambda(\mathbf{p}) e^{-ip \cdot x} \\ \langle 0, I| d_\lambda(\mathbf{p}) \psi(x) |0, I\rangle &= \frac{1}{(2\pi)^{3/2} \sqrt{2\omega}} v_\lambda(\mathbf{p}) e^{-ip \cdot x} \\ \langle 0, I| \psi(x) b_\lambda^\dagger(\mathbf{p}) |0, I\rangle &= \frac{1}{(2\pi)^{3/2} \sqrt{2\omega}} u_\lambda(\mathbf{p}) e^{ip \cdot x} \\ \langle 0, I| \bar{\psi}(x) d_\lambda^\dagger(\mathbf{p}) |0, I\rangle &= \frac{1}{(2\pi)^{3/2} \sqrt{2\omega}} \bar{v}_\lambda(\mathbf{p}) e^{ip \cdot x} \end{aligned}$$

To illustrate how to use these rules consider the process in which a single particle at early time is scattered by the external potential to a single particle at late times.

$$\langle out| b_\lambda^{out}(\mathbf{q}) b_\lambda^{in\dagger}(\mathbf{p}) |in\rangle = \langle out|in\rangle_A [\delta_{\lambda'\lambda} \delta^3(\mathbf{q} - \mathbf{p}) + \langle 0, I| b_\lambda(\mathbf{q}) U_I b_\lambda^\dagger(\mathbf{p}) |0, I\rangle^c]$$

where the superscript *c* denotes the restriction to connected diagrams containing at least one vertex.

* Note that since $U = U_0 U_I$ and $U_0^{-1} \Omega^{in} U_0$ is just a numerical phase times Ω^{in} this definition does ensure that Ω^{out} is an eigenoperator of $H_0(+\infty)$.



The first diagram has the value

$$\frac{1}{(2\pi)^3 \sqrt{4\omega(\mathbf{q})\omega(\mathbf{p})}} \int d^4x e^{i(p-q)\cdot x} \bar{u}_{\lambda'}(\mathbf{q}) i q \gamma \cdot A(x) u_{\lambda}(\mathbf{p}).$$

The second diagram has two vertices and a factor of $S_F(y-x)$ between them. But notice that the second plus all the higher diagrams just amounts to replacing this S_F by $S_F(y, x; A)$, so the sum of all the diagrams but the first has the value

$$\frac{1}{(2\pi)^3 \sqrt{4\omega(\mathbf{q})\omega(\mathbf{p})}} \int d^4x d^4y e^{i(p\cdot x - q\cdot y)} \bar{u}_{\lambda'}(\mathbf{q}) i q \gamma \cdot A(y) S_F(y, x; A) i q \gamma \cdot A(x) u_{\lambda}(\mathbf{p}).$$

We put all this together in the form

$$\langle out | b_{\lambda}^{out}(\mathbf{q}) b_{\lambda}^{in\dagger}(\mathbf{p}) | in \rangle = \langle out | in \rangle_A [\delta_{\lambda'\lambda} \delta^3(\mathbf{q} - \mathbf{p}) + \mathcal{M}(\mathbf{q}, \mathbf{p}; A)],$$

where

$$\begin{aligned} \mathcal{M}(\mathbf{q}, \mathbf{p}; A) = & \frac{1}{(2\pi)^3 \sqrt{4\omega(\mathbf{q})\omega(\mathbf{p})}} \int d^4x d^4y e^{i(p\cdot x - q\cdot y)} \\ & \bar{u}_{\lambda'}(\mathbf{q}) [i q \gamma \cdot A(x) \delta(x-y) + i q \gamma \cdot A(y) S_F(y, x; A) i q \gamma \cdot A(x)] u_{\lambda}(\mathbf{p}). \end{aligned} \quad (15.1)$$

We can put this last formula in a more suggestive form by defining the free Dirac plane wave functions

$$\psi_{\mathbf{p}\lambda}^0(x) \equiv \frac{1}{(2\pi)^{3/2} \sqrt{2\omega}} u_{\lambda}(\mathbf{p}) e^{ip\cdot x}$$

and noting that

$$\psi_{\mathbf{p}\lambda}(x) \equiv \psi_{\mathbf{p}\lambda}^0(x) + \int d^4y S_F(x, y; A) i q \gamma \cdot A(y) \psi_{\mathbf{p}\lambda}^0(y)$$

is a solution of the Dirac equation in the presence of A :

$$(m + \frac{1}{i} \gamma \cdot D) \psi_{\mathbf{p}\lambda} = 0,$$

with the boundary condition that at early times the only positive frequency components are contained in the term $\psi_{\mathbf{p}\lambda}^0(x)$. Thus we can write

$$\mathcal{M}(\mathbf{q}, \mathbf{p}; A) = \int d^4x \bar{\psi}_{\mathbf{q}\lambda'}^0(x) i q \gamma \cdot A(x) \psi_{\mathbf{p}\lambda}(x),$$

which is reminiscent of the corresponding formula in nonrelativistic quantum mechanics for the scattering of a particle from an external potential.

In the case of a static potential, the time dependence of both $\psi_{\mathbf{p}\lambda}^0(x)$ and $\psi_{\mathbf{p}\lambda}(x)$ is given by a multiplicative phase $e^{-i\omega(\mathbf{p})t}$ and the time integral then provides an energy conserving delta function

$$\mathcal{M}(\mathbf{q}, \mathbf{p}; A(\mathbf{x})) = 2\pi\delta(\omega(\mathbf{q}) - \omega(\mathbf{p})) \int d^3x \bar{\psi}_{\mathbf{q}\lambda'}^0(\mathbf{x}) i q \gamma \cdot A(\mathbf{x}) \psi_{\mathbf{p}\lambda}(\mathbf{x}).$$

Recall from basic scattering theory that if the scattering matrix for a particle from a static potential is written

$$\langle \mathbf{q}, out | \mathbf{p}, in \rangle = \delta_{\lambda'\lambda} \delta(\mathbf{q} - \mathbf{p}) - 2\pi i \delta(\omega(\mathbf{q}) - \omega(\mathbf{p})) T_{\lambda'\lambda}(\mathbf{q}, \mathbf{p})$$

then the differential scattering cross section is given by

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{d^3q}{d\Omega} \delta(\omega(\mathbf{q}) - \omega(\mathbf{p})) \frac{(2\pi)^4}{v} |T_{\lambda'\lambda}(\mathbf{q}, \mathbf{p})|^2, \\ &= q^2 dq \delta(\omega(\mathbf{q}) - \omega(\mathbf{p})) \frac{(2\pi)^4}{v} |T_{\lambda'\lambda}(\mathbf{q}, \mathbf{p})|^2, \\ &= p\omega(\mathbf{p}) \frac{(2\pi)^4}{v} |T_{\lambda'\lambda}(\mathbf{q}, \mathbf{p})|^2, \\ &= \omega(\mathbf{p})^2 (2\pi)^4 |T_{\lambda'\lambda}(\mathbf{q}, \mathbf{p})|^2, \end{aligned}$$

where v is the speed of the incident particle. Comparing with our expression we find that for electron scattering from a static potential

$$T_{\lambda'\lambda}(\mathbf{q}, \mathbf{p}) = i \int d^3x \bar{\psi}_{\mathbf{q}\lambda'}^0(\mathbf{x}) i q \gamma \cdot A(\mathbf{x}) \psi_{\mathbf{p}\lambda}(\mathbf{x}).$$

Note that in the absence of pair production, the factor $\langle out | in \rangle$, the vacuum persistence amplitude, is a pure phase and doesn't contribute in the absolute square of T . In lowest order in A (the Born approximation) one simply replaces ψ by ψ^0 .

Let us note some tricks that are useful in calculating cross sections for processes with unobserved final spins and unpolarized beams. When we calculate the absolute square of a spinor matrix element we can make use of the identities

$$\begin{aligned} \sum_{\lambda} u_{\lambda} \bar{u}_{\lambda} &= m - \gamma \cdot p \\ \sum_{\lambda} v_{\lambda} \bar{v}_{\lambda} &= -m - \gamma \cdot p. \end{aligned}$$

For example these allow us to write

$$\begin{aligned} \sum_{\lambda\lambda'} |\bar{u}_{\lambda'} \gamma \cdot \tilde{A}(\mathbf{q} - \mathbf{p}) u_{\lambda}(\mathbf{p})|^2 &= \text{Tr}[\gamma \cdot \tilde{A}(m - \gamma \cdot p) \gamma \cdot \tilde{A}^*(m - \gamma \cdot q)] \\ &= 4(-m^2 - p \cdot q) \tilde{A} \cdot \tilde{A}^* + 4(p \cdot \tilde{A} q \cdot \tilde{A}^* + q \cdot \tilde{A} p \cdot \tilde{A}^*) \\ &= 2(p - q)^2 \tilde{A} \cdot \tilde{A}^* + 4(p \cdot \tilde{A} q \cdot \tilde{A}^* + q \cdot \tilde{A} p \cdot \tilde{A}^*) \end{aligned}$$

Physical quantities should be gauge invariant and cross sections are no exception. We can reveal the gauge invariance of this last formula by introducing $\tilde{F}_{\mu\nu}(q) \equiv i(q_{\mu} \tilde{A}_{\nu} - q_{\nu} \tilde{A}_{\mu})$ which is the Fourier transform of the gauge invariant field strength. Then

$$\tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu*} = 2(q^2 \tilde{A} \cdot \tilde{A}^* - q \cdot \tilde{A} q \cdot \tilde{A}^*).$$

Recalling that the argument of \tilde{A} is $q - p$ we then find that the squared spinor matrix element can be

rewritten

$$\sum_{\lambda\lambda'} |\bar{u}_{\lambda'} \gamma \cdot \tilde{A}(\mathbf{q} - \mathbf{p}) u_{\lambda}(\mathbf{p})|^2 = \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu*} + 2(p + q) \cdot \tilde{A} (p + q) \cdot \tilde{A}^*.$$

The first term is manifestly gauge invariant and the second is gauge invariant by virtue of the identity

$$(q + p) \cdot (q - p) = q^2 - p^2 = -m^2 + m^2 = 0.$$

The differential cross section for unpolarized electron scattering in a general static potential with final spin unobserved is then

$$\begin{aligned} \sum_{\lambda'} \frac{d\sigma}{d\Omega}^{Born}_{Unpolarized} &= \frac{1}{16\pi^2} \frac{1}{2} \sum_{\lambda'\lambda} |\bar{u}_{\lambda'} q \gamma \cdot \tilde{A}(\mathbf{q} - \mathbf{p}) u_{\lambda}(\mathbf{p})|^2 \\ &= \frac{e^2}{16\pi^2} \left(\frac{1}{2} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu*} + (p + q) \cdot \tilde{A} (p + q) \cdot \tilde{A}^* \right) \\ &= \frac{\alpha}{4\pi} \left(\frac{1}{2} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu*} + |(p + q) \cdot \tilde{A}|^2 \right). \end{aligned}$$

One can easily confirm that this agrees with our previous results for the special case of the Coulomb potential.

We expect that the transition amplitudes for antiparticles should be obtained from (15.1) by the substitution $A \rightarrow -A$. On the other hand the rules seem to give a different prescription:

$$\langle out | d_{\lambda}^{out}(\mathbf{q}) d_{\lambda'}^{in\dagger}(\mathbf{p}) | in \rangle = \langle out | in \rangle_A [\delta_{\lambda'\lambda} \delta^3(\mathbf{q} - \mathbf{p}) + \bar{\mathcal{M}}(\mathbf{q}, \mathbf{p}; A)],$$

with

$$\begin{aligned} \bar{\mathcal{M}}(\mathbf{q}, \mathbf{p}; A) &= - \frac{1}{(2\pi)^3 \sqrt{4\omega(\mathbf{q})\omega(\mathbf{p})}} \int d^4x d^4y e^{i(p \cdot x - q \cdot y)} \\ &\quad \bar{v}_{\lambda}(\mathbf{p}) [iq\gamma \cdot A(x) \delta(x - y) + iq\gamma \cdot A(x) S_F(x, y; A) iq\gamma \cdot A(y)] v_{\lambda'}(\mathbf{q}). \end{aligned} \quad (15.2)$$

In fact it is not hard to show that $\bar{\mathcal{M}}(\mathbf{q}, \mathbf{p}; A) = \mathcal{M}(\mathbf{q}, \mathbf{p}; -A)$ by inserting $v = i\gamma^2 u^*$ into (15.2) and transposing the matrix element, using $i\gamma^2 \gamma^0 \gamma^{\mu T} \gamma^0 i\gamma^2 = -\gamma^{\mu}$ and $i\gamma^2 \gamma^0 S^T(x, y; A) \gamma^0 i\gamma^2 = -\gamma^{\mu} = S_F(y, x; -A)$. The latter fact can be seen either term by term in the expansion in A or by examining the defining Green function equation for $S_F(x, y; A)$.

In addition to the scattering of electrons or positrons, an external field can induce electron positron pair production and also pair annihilation. Taking the first case, for example, we have

$$\langle out | b_{\lambda}^{out}(\mathbf{q}_1) d_{\lambda'}^{out}(\mathbf{q}_2) | in \rangle = \langle out | in \rangle_A [\mathcal{M}^{PairCreate}(\mathbf{q}_1, \mathbf{q}_2; A)],$$

with

$$\begin{aligned} \mathcal{M}^{PairCreate}(\mathbf{q}_1, \mathbf{q}_2; A) &= - \frac{1}{(2\pi)^3 \sqrt{4\omega(\mathbf{q}_1)\omega(\mathbf{q}_2)}} \int d^4x d^4y e^{-i(q_1 \cdot x + q_2 \cdot y)} \\ &\quad \bar{u}_{\lambda}(\mathbf{q}_1) [iq\gamma \cdot A(x) \delta(x - y) + iq\gamma \cdot A(x) S_F(x, y; A) iq\gamma \cdot A(y)] v_{\lambda'}(\mathbf{q}_2). \end{aligned} \quad (15.3)$$

In this case, of course, there is no delta function term representing only vacuum persistence. Similarly for pair annihilation, we have

$$\langle out | b_{\lambda}^{in\dagger}(\mathbf{q}_1) d_{\lambda'}^{in\dagger}(\mathbf{q}_2) | in \rangle = \langle out | in \rangle_A [\mathcal{M}^{PairAnnih}(\mathbf{q}_1, \mathbf{q}_2; A)],$$

with

$$\begin{aligned} \mathcal{M}^{PairAnnih}(\mathbf{q}_1, \mathbf{q}_2; A) &= \frac{1}{(2\pi)^3 \sqrt{4\omega(\mathbf{q}_1)\omega(\mathbf{q}_2)}} \int d^4x d^4y e^{-i(q_2 \cdot x + q_1 \cdot y)} \\ &\quad \bar{v}_{\lambda}(\mathbf{q}_2) [iq\gamma \cdot A(x) \delta(x - y) + iq\gamma \cdot A(x) S_F(x, y; A) iq\gamma \cdot A(y)] u_{\lambda'}(\mathbf{q}_1). \end{aligned} \quad (15.3)$$

Notice the prominent appearance of the Green function $S_F(x, y; A)$ for the Dirac Equation with an

external field in all of the four basic processes, electron scattering, positron scattering, pair production, and pair annihilation. Moreover, we have also seen how to express the vacuum persistence amplitude in terms of this same Green function. Thus we see that the solutions of the first quantized Dirac equation are of direct utility in finding the physical properties of the second quantized theory.

As a final note we show how S_F even contains information about the bound states of an electron in a static external field. Actually it is better to deal not with an exactly static field, but with a field that is adiabatically switched on at some very early time $-T$, static for a very long time interval $\approx 2T$ and then adiabatically switched off at a very late time T . Then according to the adiabatic theorem, $|in\rangle$ is proportional to the ground state of the Hamiltonian $H(t)$ at all times, and $|out\rangle = |in\rangle \langle out|in\rangle$. In particular, $|in\rangle$ is the ground state $|G\rangle$ of H_A the hamiltonian for the static potential under study, since $H(t) = H_A$ for $-T \ll t \ll T$. Now fix the times $T \gg x^0 > y^0 \gg -T$. Then we have

$$\begin{aligned} S_F(x, y) \langle out|in\rangle &= \langle out| \psi(x) \bar{\psi}(y) |in\rangle \\ &= \langle out|in\rangle \langle G| \psi(x) \bar{\psi}(y) |G\rangle \\ &= \langle out|in\rangle \sum_n \langle G| \psi(x) |n\rangle \langle n| \bar{\psi}(y) |G\rangle \\ &= \langle out|in\rangle \sum_n e^{-i(E_n - E_G)(x^0 - y^0)} \langle G| \psi(\mathbf{x}, 0) |n\rangle \langle n| \bar{\psi}(\mathbf{y}, 0) |G\rangle \end{aligned}$$

From this last formula, we see that a harmonic analysis of S_F in the time variable $t = x^0 - y^0$ yields the possible energy eigenvalues $E_n - E_G$ of the energy eigenstates $|n\rangle$. These states have the quantum numbers of a one electron state because they are created from the vacuum by $\bar{\psi}$. Note that the harmonic components for $x^0 < y^0$ have the interpretation as $-(E_{\bar{n}} - E_G)$ where $E_{\bar{n}}$ are the energy levels of one positron states. Because S_F satisfies the homogeneous Dirac equation for $t \neq 0$, the possible energy eigenvalues are solutions of the time independent Dirac equation

$$(m + \frac{1}{i} \boldsymbol{\gamma} \cdot \mathbf{D} + qA^0 \gamma^0 - (E_n - E_G))\psi = 0, \quad t > 0$$

or

$$(m + \frac{1}{i} \boldsymbol{\gamma} \cdot \mathbf{D} + qA^0 \gamma^0 + (E_{\bar{n}} - E_G))\psi = 0, \quad t < 0.$$

Thus, again, the solution of the energy eigenvalue problem for the first quantized theory is directly applicable to that for the second quantized theory.

Relation to Time Ordered Products. (This subsection can be skipped in a first reading.) It is useful to establish how the scattering amplitudes are related to matrix elements of time ordered products. First let us define

$$\begin{aligned} b_\lambda^\dagger(\mathbf{p}, t) &= \int d^3x \frac{1}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{p})}} e^{ip \cdot x} \bar{\psi}(x) \gamma^0 u_\lambda(\mathbf{p}) \\ d_\lambda^\dagger(\mathbf{p}, t) &= \int d^3x \frac{1}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{p})}} e^{ip \cdot x} \bar{v}_\lambda(\mathbf{p}) \gamma^0 \psi(x) \\ b_\lambda(\mathbf{p}, t) &= \int d^3x \frac{1}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{p})}} e^{-ip \cdot x} \bar{u}_\lambda(\mathbf{p}) \gamma^0 \psi(x) \\ d_\lambda(\mathbf{p}, t) &= \int d^3x \frac{1}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{p})}} e^{-ip \cdot x} \bar{\psi}(x) \gamma^0 v_\lambda(\mathbf{p}). \end{aligned} \tag{15.4}$$

For free fields all these operators are constant in time and are just the creation and annihilation operators of the Dirac field. For a Dirac field in the presence of external fields which vanish at early and late times they are *not* constant but approach the “out” creation and annihilation operators at $t = \infty$ and the “in” operators at $t = -\infty$.

Now we shall make use of the following “reduction” trick:

$$F(t = +\infty)T[\phi_1(x_1) \cdots \phi_N(x_N)] \mp T[\phi_1(x_1) \cdots \phi_N(x_N)]F(t = -\infty) = \int_{-\infty}^{\infty} dt \frac{\partial}{\partial t} T[F(t)\phi_1(x_1) \cdots \phi_N(x_N)]$$

where $F(t)$ is any function of Heisenberg operators at time t , and the ϕ 's are generic field operators. The sign choice is + for $F(t)$ and $T[\cdots]$ fermionic operators, but $-$ in all other cases. In particular, F can be any of the expressions (15.4) where the time appearing on the r.h.s. is set to t . This leads to a series of formulae for the commutation of creation and annihilation operators with time ordered products. For instance, take F to be the r.h.s. of the expression for $b_\lambda(\mathbf{p})$.

$$\begin{aligned} b_\lambda^{out}(\mathbf{p})T[\phi_1(x_1) \cdots \phi_N(x_N)] \mp T[\phi_1(x_1) \cdots \phi_N(x_N)]b_\lambda^{in}(\mathbf{p}) &= \\ \frac{\bar{u}_\lambda(\mathbf{p})}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p})}} \int d^4x \gamma^0 \frac{\partial}{\partial t} (e^{-ip \cdot x} T[\psi(x)\phi_1(x_1) \cdots \phi_N(x_N)]) &= \\ = \frac{\bar{u}_\lambda(\mathbf{p})}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p})}} \int d^4x e^{-ip \cdot x} \left(\gamma^0 \frac{\partial}{\partial t} + i\gamma^0 \omega(\mathbf{p}) \right) T[\psi(x)\phi_1(x_1) \cdots \phi_N(x_N)] &= \\ = \frac{\bar{u}_\lambda(\mathbf{p})}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p})}} \int d^4x e^{-ip \cdot x} i(m - i\gamma \cdot \partial) T[\psi(x)\phi_1(x_1) \cdots \phi_N(x_N)] \end{aligned}$$

where in the last line we used

$$\bar{u}\gamma^0\omega e^{-ip \cdot x} = \bar{u}(m + i\boldsymbol{\gamma} \cdot \nabla)e^{-ip \cdot x}$$

and then integrated by parts. Clearly this derivation can be repeated for each of the expressions in (15.4) leading to the reduction formulae:

$$\begin{aligned} b_\lambda^{out}(\mathbf{p})T[\phi_1(x_1) \cdots \phi_N(x_N)] \mp T[\phi_1(x_1) \cdots \phi_N(x_N)]b_\lambda^{in}(\mathbf{p}) &= \\ \frac{\bar{u}_\lambda(\mathbf{p})}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p})}} \int d^4x e^{-ip \cdot x} i(m - i\boldsymbol{\gamma} \cdot \partial) T[\psi(x)\phi_1(x_1) \cdots \phi_N(x_N)] &= \\ b_\lambda^{out\dagger}(\mathbf{p})T[\phi_1(x_1) \cdots \phi_N(x_N)] \mp T[\phi_1(x_1) \cdots \phi_N(x_N)]b_\lambda^{in\dagger}(\mathbf{p}) &= \\ \int d^4x T[\bar{\psi}(x)\phi_1(x_1) \cdots \phi_N(x_N)]i \left(-m - i\boldsymbol{\gamma} \cdot \overleftarrow{\partial} \right) e^{ip \cdot x} \frac{u_\lambda(\mathbf{p})}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p})}} &= \\ d_\lambda^{out}(\mathbf{p})T[\phi_1(x_1) \cdots \phi_N(x_N)] \mp T[\phi_1(x_1) \cdots \phi_N(x_N)]d_\lambda^{in}(\mathbf{p}) &= \\ \int d^4x T[\bar{\psi}(x)\phi_1(x_1) \cdots \phi_N(x_N)]i \left(-m - i\boldsymbol{\gamma} \cdot \overleftarrow{\partial} \right) e^{-ip \cdot x} \frac{v_\lambda(\mathbf{p})}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p})}} &= \\ d_\lambda^{out\dagger}(\mathbf{p})T[\phi_1(x_1) \cdots \phi_N(x_N)] \mp T[\phi_1(x_1) \cdots \phi_N(x_N)]d_\lambda^{in\dagger}(\mathbf{p}) &= \\ \frac{\bar{v}_\lambda(\mathbf{p})}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p})}} \int d^4x e^{ip \cdot x} i(m - i\boldsymbol{\gamma} \cdot \partial) T[\psi(x)\phi_1(x_1) \cdots \phi_N(x_N)] \end{aligned} \tag{15.5}$$

The reduction formulae can be used to systematically reduce scattering amplitudes to expressions directly involving time ordered products.

We shall illustrate the procedure for the case of a particle scattering in an external field. The application

to antiparticle scattering and pair production and annihilation will be left to the reader:

$$\begin{aligned}
\langle out| b_{\lambda'}^{out}(\mathbf{q}) b_{\lambda}^{in\dagger}(\mathbf{p}) |in\rangle &= \langle out| b_{\lambda'}^{in}(\mathbf{q}) b_{\lambda}^{in\dagger}(\mathbf{p}) |in\rangle \\
&+ \frac{\bar{u}_{\lambda'}(\mathbf{q})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{q})}} \int d^4x e^{-iq \cdot x} i(m - i\gamma \cdot \partial) \langle out| \psi(x) b_{\lambda}^{in\dagger}(\mathbf{p}) |in\rangle \\
&= \langle out|in\rangle \delta_{\lambda\lambda'} \delta(\mathbf{q} - \mathbf{p}) + \frac{\bar{u}_{\lambda'}(\mathbf{q})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{q})}} \int d^4x d^4y \\
e^{-iq \cdot x} i(m - i\gamma \cdot \partial) \langle out| T[\bar{\psi}(y)\psi(x)] |in\rangle i(-m - i\gamma \cdot \overleftarrow{\partial}_y) e^{ip \cdot y} \frac{u_{\lambda}(\mathbf{p})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{p})}} \\
&= \langle out|in\rangle \left[\delta_{\lambda\lambda'} \delta(\mathbf{q} - \mathbf{p}) + \frac{\bar{u}_{\lambda'}(\mathbf{q})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{q})}} \int d^4x d^4y \right. \\
&\left. e^{-iq \cdot x} i(m - i\gamma \cdot \partial) S_F(x, y; A) i(m + i\gamma \cdot \overleftarrow{\partial}_y) e^{ip \cdot y} \frac{u_{\lambda}(\mathbf{p})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{p})}} \right]
\end{aligned}$$

This is identical to our original expression as can be seen by writing $m - i\gamma \cdot \partial = m - i\gamma \cdot D + q\gamma \cdot A$ so that

$$i(m - i\gamma \cdot \partial) S_F(x, y; A) i(m + i\gamma \cdot \overleftarrow{\partial}_y) = \delta(x - y) i(m + i\gamma \cdot \overleftarrow{\partial}_y) + iq\gamma \cdot A S_F(x, y; A) i(m + i\gamma \cdot \overleftarrow{\partial}_y).$$

The first term on the r.h.s contributes a term $\bar{u}(q)(m + \gamma \cdot q)u(p) = 0$ by the Dirac equation. The second term can be simplified using the fact that S is also a Green function in its second argument:

$$S_F(m + i\gamma \cdot \overleftarrow{\partial}_y - q\gamma \cdot A) = -i\delta(x - y).$$

Thus we have

$$i(m - i\gamma \cdot \partial) S_F(x, y; A) i(m + i\gamma \cdot \overleftarrow{\partial}_y) = iq\gamma \cdot A \delta(x - y) + iq\gamma \cdot A S_F(x, y; A) iq\gamma \cdot A$$

as desired.

Now let us consider a little more closely the meaning of the reduction formula

$$\begin{aligned}
\langle out| b_{\lambda'}^{out}(\mathbf{q}) b_{\lambda}^{in\dagger}(\mathbf{p}) |in\rangle &= \langle out|in\rangle \left[\delta_{\lambda\lambda'} \delta(\mathbf{q} - \mathbf{p}) + \frac{\bar{u}_{\lambda'}(\mathbf{q})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{q})}} \int d^4x d^4y \right. \\
&\left. e^{-iq \cdot x} i(m - i\gamma \cdot \partial) S_F(x, y; A) i(m + i\gamma \cdot \overleftarrow{\partial}_y) e^{ip \cdot y} \frac{u_{\lambda}(\mathbf{p})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{p})}} \right].
\end{aligned} \tag{15.6}$$

The factors $(m - i\gamma \cdot \partial)$ and $i(m + i\gamma \cdot \overleftarrow{\partial}_y)$ look as though they should give zero because after integrating by parts, they become $(m + \gamma \cdot q)$ and $(m + \gamma \cdot p)$ respectively and these factors give zero next to the Dirac spinors. The error in this reasoning is of course that the surface terms at $t = \pm\infty$ are not zero (being in fact the scattering amplitudes themselves!). To get a clearer idea of what is happening, suppose we continue q^0, p^0 a little away from their “on-shell” values of $\omega(\mathbf{q}), \omega(\mathbf{p})$. Then the surface terms oscillate at infinity and are effectively zero (*e.g.* in any smooth wave packet). Then we can integrate by parts and the second term becomes

$$\begin{aligned}
&\frac{\bar{u}_{\lambda'}(\mathbf{q})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{q})}} i(m + \gamma \cdot q) \int d^4x d^4y e^{-iq \cdot x + ip \cdot y} S_F(x, y; A) i(m + \gamma \cdot p) \frac{u_{\lambda}(\mathbf{p})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{p})}} \\
&\equiv \frac{\bar{u}_{\lambda'}(\mathbf{q})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{q})}} i(m + \gamma \cdot q) T(q, p) i(m + \gamma \cdot p) \frac{u_{\lambda}(\mathbf{p})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{p})}}.
\end{aligned} \tag{15.7}$$

where the r.h.s. defines $T(q, p)$, the Fourier transform of the Green function. The only way for this to be nonzero as q^0, p^0 approach their on-shell values is for T to acquire poles in this limit. The residues of these poles are then related to the scattering amplitudes. This is the content of the reduction formula, but there is a rather direct way to see how these poles come about.

Consider first of all the region of integration $x^0 > T$ where ψ is a free field, with $b^{out}, d^{out\dagger}$. Then this region contributes to $T(q, p)$ the bit

$$\int_T^\infty dt e^{i(q^0 - \omega(\mathbf{q}))t} \frac{(2\pi)^{3/2}}{\sqrt{2\omega(\mathbf{q})}} \langle out | \sum_{\lambda'} b^{out}(\mathbf{q}) u_{\lambda'}(\mathbf{q}) \bar{\psi}(y) | in \rangle$$

For $q^0 = \omega$ the integrand is time independent so the integral over an infinite range gives a divergence. To study it, give q^0 a small positive imaginary part and do the integral:

$$\frac{i e^{i(q^0 - \omega(\mathbf{q}))T}}{q^0 - \omega(\mathbf{q}) + i\epsilon} \frac{(2\pi)^{3/2}}{\sqrt{2\omega(\mathbf{q})}} \langle out | \sum_{\lambda'} b^{out}(\mathbf{q}) u_{\lambda'}(\mathbf{q}) \bar{\psi}(y) | in \rangle$$

For the range of integration with $x^0 < -T$ ψ contains $b^{in}, d^{in\dagger}$ and stands on the right so only the second operator contributes. The integral over this range is not singular for $q^0 \rightarrow \omega$.

Similar considerations apply to the integral over y^0 . This time the region $y^0 < -T$ contributes a pole at $p^0 = \omega(\mathbf{p})$ involving $b^{in\dagger}$. There is also a pole from the region $y^0 > T$ involving $b^{out\dagger}$. This contribution doesn't vanish because of the presence of b^{out} from the first reduction; it just gives delta functions. Going through all these steps leads to

$$T(q, p) \underset{q^0, p^0 \rightarrow \omega(\mathbf{q}), \omega(\mathbf{p})}{\sim} \frac{i}{q^0 - \omega(\mathbf{q})} \frac{i}{p^0 - \omega(\mathbf{p})} \left[\frac{(2\pi)^3}{\sqrt{4\omega(\mathbf{q})\omega(\mathbf{p})}} \right] \sum_{\lambda', \lambda} u_{\lambda'}(\mathbf{q}) \bar{u}_\lambda(\mathbf{p}) \\ [\langle out | b_{\lambda'}^{out}(\mathbf{q}) b_\lambda^{in\dagger}(\mathbf{p}) | in \rangle - \langle out | in \rangle \delta_{\lambda', \lambda} \delta(\mathbf{q} - \mathbf{p})]$$

which is of course exactly the behavior required to satisfy the reduction formula.

16. Path Integrals for Gauge Fields

The path integral for gauge theories presents special problems because of the gauge invariance of the action. We shall establish the proper formulation for Quantum Electrodynamics by first obtaining the path integral in Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, following the general procedure sketched in this chapter. Once that has been done, we can discuss within the new path integral formalism more general gauges, including covariant ones.

The Hamiltonian operator for charged fields interacting with the quantized e.m. field \mathbf{A} in Coulomb gauge is (as we shall see in the next chapter)

$$\begin{aligned}
 H_{eff} = \int d^3x \left(\frac{1}{2} \mathbf{\Pi}_T^2 + \frac{1}{2} \partial_k \mathbf{A}_T \cdot \partial_k \mathbf{A}_T - \mathbf{A}_T \cdot \mathbf{J}_e \right) \\
 + \int (J_e^0 + j^0) \left(-\frac{1}{2\nabla^2} \right) (J_e^0 + j^0) + H_{fields}(\mathbf{A})|_{A_0=0},
 \end{aligned} \tag{6.6}$$

Here we understand all operators to be in Schrödinger picture. We see that this is a case where the conjugate momentum appears only quadratically so we can immediately transcribe an infinitesimal evolution between eigenstates of \mathbf{A} near imaginary time τ

$$\begin{aligned}
 \langle \mathbf{A}'' | e^{-d\tau H_{eff}} | \mathbf{A}' \rangle \\
 \approx \delta(\nabla \cdot \mathbf{A}'') \exp \left\{ d\tau \int d^3x \left[\frac{1}{2} (\dot{\mathbf{A}})^2(\tau) + \frac{1}{2} (\partial \mathbf{A})^2(\tau) - \mathbf{A}(\tau) \cdot \mathbf{J}_e \right] \right. \\
 \left. + d\tau \int (J_e^0 + j^0) \left(-\frac{1}{2\nabla^2} \right) (J_e^0 + j^0) + d\tau H_{fields}(\mathbf{A}(\tau))|_{A_0=0} \right\},
 \end{aligned}$$

where the terms involving $\mathbf{A}(\tau)$ with no time derivative are averaged over $\mathbf{A}(\tau) = \mathbf{A}''$, \mathbf{A}' and $\dot{\mathbf{A}}(\tau)$ symbolizes the continuum limit of $(\mathbf{A}'' - \mathbf{A}')/d\tau$. The delta function multiplying the r.h.s. symbolizes the condition $\nabla \cdot \mathbf{A} = 0$. It is necessary if we wish to formally integrate over all three components of \mathbf{A} . Notice that for the moment we are only changing the e.m. field into path integral language, leaving the charged fields as operators.

Next we employ a widely used trick for simplifying the term describing the instantaneous Coulomb interaction. This involves introducing an auxiliary variable $A^4(\mathbf{x}, t)$ and writing the identity

$$e^{-d\tau \int (J_e^0 + j^0) \left(-\frac{1}{2\nabla^2} \right) (J_e^0 + j^0)} = \frac{\int \mathcal{D}A^4(\tau) \exp \left\{ -d\tau \int d^3x \left[\frac{1}{2} (\nabla A^4)^2 - A^4 (J_e^0 + j^0) \right] \right\}}{\int \mathcal{D}A^4(\tau) \exp \left\{ -d\tau \int d^3x \frac{1}{2} (\nabla A^4)^2 \right\}}$$

where $j^4 = ij^0$ and $J_e^4 = iJ_e^0$ as appropriate after the Wick rotation. This identity is proved by completing the square by a shift of integration variable $A^4 \rightarrow A^4 - (1/\nabla^2)j^4$. Here we use the fact that ultimately $d\tau \rightarrow 0$ to neglect any commutators that might arise due to the fact that j^0 is really an operator. (This latter approximation is in exactly the same spirit as the replacement of \mathcal{H}_k by H_k^W in our general derivation of the path integral.) The variable A^4 enters now exactly as the imaginary time component of the vector potential would have entered before it was eliminated by solving the Gauss' law constraint. For example, using $\nabla \cdot \mathbf{A} = 0$ we can write

$$\int d^3x \left[\frac{1}{2} \dot{\mathbf{A}}^2 + \frac{1}{2} (\nabla \mathbf{A}_k)^2 - \mathbf{A} \cdot \mathbf{J}_e \right] + \frac{1}{2} (\nabla A^4)^2 - A^4 J_e^4 = \int d^3x \left[\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - A_\mu J_e^\mu \right].$$

Furthermore $H_{fields}|_{A^4=0} - \int d^3x A^4(\tau) j^4 = H_{fields}$ with $A^4 \neq 0$ playing the role of the imaginary time component of the vector potential.

Composing many infinitesimal evolutions, we arrive at the path integral representation

$$\langle \mathbf{A}_f | U(\infty, -\infty) | \mathbf{A}_i \rangle = \frac{\int \mathcal{D}A_\mu \delta(\nabla \cdot \mathbf{A}) e^{-\int d^4x (F_{\mu\nu} F_{\mu\nu}/4 - A_\mu J_e^\mu)} \langle f | T e^{-\int d\tau H_{fields}(A_\mu(\tau))} | i \rangle}{\int \mathcal{D}A^4 e^{-\int d\tau (\nabla A^4)^2/2}}.$$

As discussed before, by taking the evolution over an infinite time interval we effectively project onto the ground states (vacuum) at early and late times. We can therefore pick convenient initial and final states, *e.g.* we can let them be eigenstates of \mathbf{A} with vanishing eigenvalue and ground states of $H_{fields}(A_\mu(\tau = \mp\infty), \mp\infty)$ respectively. We then obtain the following general formula for the *outin* matrix element of time ordered products of fields

$$\langle out | T[\psi(y_1) \cdots \psi(y_m) \bar{\psi}(z_m) \cdots \bar{\psi}(z_1)] | in \rangle_{J_e, A_e} = \frac{\int \mathcal{D}A_\mu \delta(\nabla \cdot \mathbf{A}) e^{-\int d^4x (F_{\mu\nu} F_{\mu\nu}/4 - A_\mu J_e^\mu)} \langle out | T[\psi(y_1) \cdots \bar{\psi}(z_1)] | in \rangle_{A(\tau) + A_e(\tau)}}{\int \mathcal{D}A_\mu \delta(\nabla \cdot \mathbf{A}) e^{-\int d^4x F_{\mu\nu} F_{\mu\nu}/4} \langle out | in \rangle_{A(\tau)}}.$$

An important observation is that the set of operators in the time ordered product on the l.h.s. can be expanded to include any number of vector potentials by functionally differentiating w.r.t. the external current J_e^μ . This is clear from the pure exponential dependence on J_e on the r.h.s. Thus in addition to describing the presence of real external sources, the J_e dependence provides a generating function for all correlation functions of any number of vector potentials in the source free case: simply set $J_e = 0$ *after* differentiating the appropriate number of times w.r.t. J_e . To illustrate this point, consider the path integral for the free e.m. field (no charged fields).

$$\langle out | in \rangle_{J_e} = \frac{\int \mathcal{D}A_\mu \delta(\nabla \cdot \mathbf{A}) e^{-\int d^4x (F_{\mu\nu} F_{\mu\nu}/4 - A_\mu J_e^\mu)}}{\int \mathcal{D}A_\mu \delta(\nabla \cdot \mathbf{A}) e^{-\int d^4x F_{\mu\nu} F_{\mu\nu}/4}}.$$

We can extract the dependence on J_e by shifting the integration variable in the numerator by $A_\mu \rightarrow A_\mu + C_\mu$, where C is chosen so that the linear term in A is cancelled. It must of course also be restricted by the Coulomb gauge condition $\nabla \cdot \mathbf{C} = 0$. After the shift the coefficient of the linear term in A becomes after an integration by parts

$$\partial_\mu (\partial_\mu C_\nu - \partial_\nu C_\mu) + J_\nu = 0.$$

This equation is only consistent if the external current is conserved $\partial_\mu J^\mu = 0$ which we are assuming. For $\nu = 4$, remembering the Coulomb gauge condition, this equation determines $C^4 = (-1/\nabla^2) J^4$. After using current conservation to write $\partial_4 C^4 = (1/\nabla^2) \nabla \cdot \mathbf{J}_e$ the spatial components are determined to be $\mathbf{C} = (-1/\partial^2)(\mathbf{J} - \nabla(1/\nabla^2) \nabla \cdot \mathbf{J})$. The question of boundary conditions is settled in Euclidean space by requiring that C vanish in all four directions at infinity. As we have discussed this prescription becomes the familiar $i\epsilon$ one when continued back to Minkowski space. Inserting these results into the path integral we find

$$\begin{aligned} \langle out | in \rangle_{J_e} &= e^{-\int d^4x (F_{\mu\nu}^C F_{\mu\nu}^C/4 - C_\mu J_e^\mu)} \\ &= e^{\int d^4x C_\mu J_e^\mu/2} \\ &= e^{\frac{1}{2} \int d^4x d^4y J_e^\mu(x) D_{\mu\nu}(x-y) J_e^\nu(y)}. \end{aligned}$$

where

$$D_{jk}(x) = \int \frac{d^4 p}{(2\pi)^4} e^{ix \cdot p} \frac{\delta_{jk} - p^j p^k / \mathbf{p}^2}{p^2}$$

$$D_{44}(x) = \int \frac{d^4 p}{(2\pi)^4} e^{ix \cdot p} \frac{1}{\mathbf{p}^2} = \delta(t) \frac{1}{4\pi|\mathbf{x}|}$$

$$D_{4k} = D_{k4} = 0.$$

To get correlation functions of any number of vector potentials we differentiate $\langle out|in \rangle_{J_e}$ with respect to the current any number of times. Because of current conservation we may only differentiate w.r.t. three components of J^μ say the two transverse components \mathbf{J}_T and J^4 . But that is sufficient since the longitudinal component of \mathbf{A} is zero by the gauge condition. If we set $J_e = 0$ after differentiating, it is easy to check that the Wick expansion follows with the two point function

$$\langle A_\mu(x) A_\nu(y) \rangle = D_{\mu\nu}(x - y).$$

Note that with path integrals it is more precise to speak of correlation functions, which are functional averages of some number of fields, than of the vacuum expectations of time ordered products of field operators. They are of course numerically equal to each other.

General Gauges The path integral formulation of gauge theories is particularly suited to the discussion of general gauges. To keep the formalism covariant we would like to be able to replace the noncovariant Coulomb gauge condition with a covariant one such as $\partial_\mu A_\mu = 0$. The procedure for gauge fixing that follows is due to Fadeev and Popov, and is quite general. The method starts by selecting some general gauge fixing condition $F(A, \partial A) = 0$. This condition should have the property that for any value of the gauge potential it is possible to find a gauge transformation to a potential for which $F = 0$, and further that if $F(A) = 0$ then $F(A_\Omega) \neq 0$ with Ω any nontrivial gauge transformation which vanishes at infinity. We don't require this property for more general gauge transformations because that would rule out Coulomb gauge which seems to be perfectly adequate. The path integrand in such a gauge should contain a factor of $\delta(F(A))$.

The F-P procedure is to define a functional $\Delta_F(A)$ by the requirement

$$1 = \Delta_F(A) \int \mathcal{D}\Omega \delta(F(A_\Omega))$$

where A_Ω is the transformation of A by the gauge group element $\Omega(x)$, and the measure $\mathcal{D}\Omega$ is gauge invariant. By this definition Δ_F is clearly gauge invariant. Now insert this representation for 1 in the "unfixed" gauge field path integrand. Next change functional integration variables so that $A_\Omega \rightarrow A$. Here we implicitly assume that the unfixed measure $\mathcal{D}A_\mu$ is invariant under changes of variables which are gauge transformations^{*}. Then the infinite volume of the gauge group $\int \mathcal{D}\Omega$ comes out as a common factor in both the numerator and denominator of the functional average and so cancels. We are left with the factors

$$\Delta_F(A) \delta(F(A))$$

in the functional integrand. The delta function fixes the gauge and the factor Δ_A is in general needed to guarantee that different choices for F yield the same answer for gauge invariant quantities. (It is only for functional averages of gauge invariant quantities that the rest of the integrand stays invariant under the variable change that removes Ω from the delta function.)

* In a completely general context this assumption might clash with the more basic translational invariance of the measure. In such a case there would be a Jacobian accompanying the variable change. This is not needed for the usual abelian and nonabelian gauge theories however.

Our Coulomb gauge path integral did not include a factor of $\Delta_{Coul}(A)$ so we need to confirm that it is inconsequential for the abelian case. For this we have to compute

$$\frac{1}{\Delta_{Coul}} = \int \mathcal{D}\Lambda \delta(\nabla \cdot \mathbf{A} + \nabla^2 \Lambda) = \frac{1}{\det(-\nabla^2)}.$$

We see that Δ_{Coul} is independent of A so that it will cancel between the numerator and denominator of functional averages. Thus our failure to put it in gives no differences in physical quantities. Thus all gauge choices are formally equivalent to Coulomb gauge provided the Fadeev-Popov factor $\Delta_F(A)$ is included along with the gauge fixing delta functional.

In practice $\Delta_F(A)$ always multiplies $\delta(F(A))$ so the former is only needed for A satisfying the gauge condition. For such A , $\delta(F(A_\Omega))$ only contributes for infinitesimal Ω :

$$F(A(x) + \delta A(x)) \approx \int d^4 y \frac{\delta F(A(x))}{\delta A_{\mu ab}(y)} \delta A_{\mu ab}(y).$$

Denoting the infinitesimal generators of the gauge group by $G(z)$, δA is linear in the matrix elements of G : $\delta A_{\mu ab}(y) = \int d^4 z \mathcal{L}_{\mu ab, cd}(y, z; A) G_{cd}(z)$ where \mathcal{L} is a linear differential operator depending on A in general. For example, the infinitesimal nonabelian gauge transformation reads $\delta A = \partial G - ig[A, G]$. Thus $\Delta_F = \det(\frac{\delta F}{\delta A_\mu} \mathcal{L}_\mu)$ where the determinant is that of a linear differential operator which is also a matrix in the internal group space with matrix elements labeled (ab, ef) :

$$\int d^4 y \frac{\delta F_{ab}(A(x))}{\delta A_{\mu cd}(y)} \mathcal{L}_{\mu cd, ef}(y, z; A).$$

An important class of covariant gauges consists of the Lorentz gauges $\partial \cdot A = f$ where f is some fixed function. The F-P determinant for this gauge is $\det(-\partial^2)$ in the abelian case of QED. Notice that it is independent of both f and A . In the nonabelian case, it would depend upon A of course, but is still independent of f . So for QED the F-P determinant cancels between numerator and denominator. and it is safe to ignore it[†]. Functional averages of gauge invariant quantities will be independent of f . We can exploit this to get rid of the functional delta function in the path integrand by averaging over f with a gaussian weight function $e^{-\int d^4 x f^2(x)/2\alpha}$. This just multiplies the numerator and denominator by the same constant so it won't alter gauge invariant quantities. The net effect of this is to remove the delta function and instead add a non gauge invariant term $\int d^4 x (\partial \cdot A)^2(x)/2\alpha$ to the Euclidean action. For $\alpha = 1$ (Feynman Gauge) the effect of this term is to simplify the kinetic term for A :

$$\int d^4 x \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2\alpha} (\partial \cdot A)^2 \right) = \int d^4 x \left(\frac{1}{2} \partial_\mu A_\nu \partial_\mu A_\nu + \frac{1-\alpha}{2\alpha} (\partial \cdot A)^2 \right)$$

$$\underset{\alpha \rightarrow 1}{\rightsquigarrow} \int d^4 x \frac{1}{2} \partial_\mu A_\nu \partial_\mu A_\nu$$

Just as with the Coulomb gauge, the photon propagator for the “ α ” gauges is obtained by evaluating the path integral for the gauge field in the presence of an external source. The source dependence is easily

[†] There are some applications where it is nonetheless important to keep it. For example, it contributes a constant to the zero point energy which just subtracts the spurious contribution of the time-like and longitudinal components of the vector potential. Also when one computes the finite temperature partition function in a covariant gauge, the F-P factor removes two photon degrees of freedom so the total number is $4 - 2 = 2$. Without the F-P factor the Stefan-Boltzmann law would be off by a factor of 2!

obtained by shifting $A \rightarrow A + C$ with C satisfying

$$\partial^2 C_\mu + \frac{1-\alpha}{\alpha} \partial \partial \cdot C = -J^\mu$$

which is solved first for $\partial \cdot C = -\alpha \cdot J$ and then

$$C_\mu = -\frac{1}{\partial^2} (J_\mu - (1-\alpha) \frac{\partial_\mu \partial_\nu}{\partial^2} J^\nu).$$

Thus the propagator is

$$D_{\mu\nu}^\alpha(x) = \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot x} \frac{\delta_{\mu\nu} - (1-\alpha) p_\mu p_\nu / p^2}{p^2}.$$

In the continuation to Minkowski space $\delta_{\mu\nu}$ is replaced by $\eta_{\mu\nu}$, p^2 by $p^2 - i\epsilon$, and there is an additional factor of $-i$. We note the great simplification for Feynman gauge. The case $\alpha = 0$, known as Landau gauge is effectively the gauge $\partial \cdot A = 0$ because the coefficient of the gauge breaking term in the action blows up damping out all contributions to the integral not satisfying this condition. We see that in this case the divergence of the propagator vanishes, $\partial_\mu D_{\mu\nu}^0 = 0$. The fundamental reason that QED is independent of α is that A always couples to a conserved current so that the terms involving α in the propagator decouple.

To get a bit more insight into the role of current conservation consider a photon propagator attached to two conserved vertices in momentum space

$$Amp \sim \frac{R_\mu^1 R^{2\mu}}{p^2} = \frac{\mathbf{R}^1 \cdot \mathbf{R}^2 - R_0^1 R_0^2}{p^2}.$$

It would seem that this coupling corresponds to the propagation of four photon states, 3 “space” components and 1 time component, the last one coupling with the “wrong” sign. These wrong sign states are sometimes called “ghosts.” But by current conservation $R_0^a = \mathbf{p} \cdot \mathbf{R}^a / p^0$. In the limit of physical photon momentum, *i.e.* for which $p^2 = 0$ the residue of the pole is just

$$\mathbf{R}^1 \cdot \mathbf{R}^2 - \frac{\mathbf{p} \cdot \mathbf{R}^1 \mathbf{p} \cdot \mathbf{R}^2}{p^2}$$

which is to say only the transverse states (perpendicular to \mathbf{p}) truly propagate.

17. Operator Quantization of the Electromagnetic Field

Treated classically the e.m. field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ satisfies Maxwell's equations

$$\partial_\nu F^{\mu\nu} = J^\mu,$$

which imply current conservation $\partial_\mu J^\mu = 0$ for consistency. So far we have developed the theory of Dirac and scalar quantum fields interacting with fixed external e.m. fields. For such systems the current is of course an operator, so Maxwell's equations imply that the e.m. field must also be a quantum operator, which inserted into the Dirac equation produces a nonlinear quantum field equation.

17.1. QUANTIZED ELECTROMAGNETIC FIELD INTERACTING WITH A CONSERVED CURRENT

The first step in developing the quantum theory for A_μ is to understand the canonical structure of Maxwell's equations. One first notes that the equations follow from stationarity of the action

$$S = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} + \int d^4x A_\mu J^\mu,$$

which is gauge invariant provided $\partial_\mu J^\mu = 0$. It is straightforward to find the momentum conjugate to A_μ

$$\Pi^\mu = F_0{}^\mu = \partial_0 A^\mu - \partial^\mu A_0.$$

The spatial components of Π are just those of minus the electric field strength, but $\Pi^0 = 0$. This last fact poses a difficulty for quantization since it is inconsistent with nonvanishing canonical commutation relations. We know how to assign operator properties to \vec{A} and $\vec{\Pi}$, but not to A_0 . Before facing this difficulty, we construct the canonical Hamiltonian

$$\begin{aligned} H_{CAN} &= \int d^3x [\vec{\Pi} \cdot \vec{A} - \mathcal{L}] \\ &= \int d^3x \left[\frac{1}{2} \vec{\Pi}^2 + \frac{1}{2} (\nabla \times \vec{A})^2 - \vec{A} \cdot \vec{J} + \nabla A_0 \cdot \vec{\Pi} - A_0 J^0 \right]. \end{aligned}$$

Notice that the troublesome variable A_0 appears only linearly and in the last two terms. After an integration by parts the coefficient is just $-(\nabla \cdot \vec{\Pi} + J^0)$, which would vanish if we could use Gauss' Law.

Classically, we could certainly enforce Gauss' Law and then the canonically uncertain A_0 would disappear from the dynamics. Attempting to enforce Gauss' Law as a quantum operator equation would contradict the canonical commutation relations for $\vec{A}, \vec{\Pi}$

$$[A_k(\vec{x}, t), \Pi^m(\vec{y}, t)] = i\delta_k^m \delta(\vec{x} - \vec{y}),$$

so we postpone discussion of this point. If we leave the operator character of A_0 unspecified, it is clearly important to know how quantum evolution will be affected if we make a change in A_0 . Let the evolution operator for a given A_0 be $U_{A_0}(t, -\infty)$. Then by a familiar argument

$$\delta U_{A_0}(t, -\infty) = i U_{A_0}(t, -\infty) \int d^4x \delta A_0 (\nabla \cdot \vec{\Pi} + J^0)$$

where the operators multiplying U_{A_0} are in Heisenberg picture. The Heisenberg equations for $\vec{A}, \vec{\Pi}$ imply

$$\begin{aligned} \frac{\partial}{\partial t} (\nabla \cdot \vec{\Pi} + J^0) &= \frac{\partial J^0}{\partial t} + \nabla \cdot \vec{J} + \int d^3x [\nabla \cdot \vec{\Pi}, A_0] (\nabla \cdot \vec{\Pi} + J^0) \\ &= i \int d^3x [\nabla \cdot \vec{\Pi}, A_0] (\nabla \cdot \vec{\Pi} + J^0) \end{aligned} \tag{17.1}$$

Although we are not free to impose Gauss' Law as an operator equation, we can restrict our incoming

states to satisfy

$$(\nabla \cdot \vec{\Pi} + J^0)|_{t=-\infty} |in\rangle = 0.$$

It then follows from (17.1) that $(\nabla \cdot \vec{\Pi} + J^0)|in\rangle = 0$ for all time. By making this restriction we therefore arrange that

$$\delta U_{A_0} |in\rangle = 0$$

for arbitrary operator changes in A_0 , that is, the quantum evolution is independent of how we treat A_0 . This is the quantum analog of what is true in the classical treatment, and suffices to resolve the difficulties. By imposing Gauss' Law on initial states, we are free to make a choice for A_0 which simplifies the dynamical problem.

For example, suppose we want to "solve" the constraint by "imposing Coulomb gauge." We do this by writing

$$\begin{aligned} \vec{\Pi} &= \vec{\Pi}_T + \frac{1}{\nabla^2} \vec{\nabla} \nabla \cdot \vec{\Pi} \\ &= \vec{\Pi}_T - \frac{1}{\nabla^2} \vec{\nabla} J^0 + \frac{1}{\nabla^2} \vec{\nabla} (\nabla \cdot \vec{\Pi} + J^0) \end{aligned} \quad (17.2)$$

so that $\nabla \cdot \vec{\Pi}_T = 0$. Inserting (17.2) into the term in the Hamiltonian containing Π and judiciously integrating by parts, we obtain

$$\begin{aligned} \int d^3x \vec{\Pi}^2 &= \int d^3x \vec{\Pi}_T^2 + \int J^0 \left(-\frac{1}{\nabla^2} \right) J^0 \\ &\quad + \int \left(2\frac{1}{\nabla^2} J^0 - \frac{1}{\nabla^2} (\nabla \cdot \vec{\Pi} + J^0) \right) (\nabla \cdot \vec{\Pi} + J^0) + \int \left[\frac{1}{\nabla^2} J^0, \nabla \cdot \vec{\Pi} + J^0 \right]. \end{aligned}$$

The last term, which arises because we reordered operators so that $\nabla \cdot \vec{\Pi} + J^0$ stands on the right wherever it appears, formally vanishes because the fields entering J are canonically independent of Π and also J^0 commutes with itself at equal times. The next to last term has a factor of $\nabla \cdot \vec{\Pi} + J^0$ on the right. By choosing

$$A_0 = \frac{1}{\nabla^2} J^0 - \frac{1}{2\nabla^2} (\nabla \cdot \vec{\Pi} + J^0),$$

this term is cancelled by the term linear in A_0 and the Hamiltonian then simplifies to

$$H_{COUL} = \int d^3x \vec{\Pi}_T^2 + \int J^0 \left(-\frac{1}{\nabla^2} \right) J^0 + \int \left(\frac{1}{2} (\nabla \times \vec{A})^2 - \vec{A} \cdot \vec{J} \right).$$

Since only Π_T now enters the Hamiltonian, it is appropriate to make a similar decomposition of $\vec{A} = \vec{A}_T + \vec{\nabla}(1/\nabla^2)\nabla \cdot \vec{A}$. The gradient term does not contribute to the curl of A and an integration by parts allows that term in H_{COUL} to be simplified as

$$\begin{aligned} \int (\nabla \times \vec{A})^2 &= \int (\nabla \times \vec{A}_T)^2 = \int \vec{A}_T \cdot (\nabla \times (\nabla \times \vec{A}_T)) \\ &= \int \vec{A}_T \cdot (-\nabla^2) \vec{A}_T = \int \nabla_k \vec{A}_T \cdot \nabla_k \vec{A}_T. \end{aligned}$$

The longitudinal component of \vec{A} appears in H_{COUL} through the coupling to \vec{J} . To interpret this term we have to consider a little more closely the inner product structure on the space of states. The states which satisfy the Gauss' Law constraint have the character of momentum eigenstates with a fixed eigenvalue. Such

states would of course have infinite norm. If we restrict our *in* states to satisfy Gauss' Law, and we want our *out* states to have a finite inner product with these *in* states, it follows that the *out* states should not be taken to satisfy the constraint. In fact, the natural dual space to momentum eigenstates are position eigenstates, with $\langle q|p\rangle \propto e^{iqp}$. Thus we should choose our *out* states to be eigenstates of $\nabla \cdot \vec{A}$, which is the conjugate variable to $\nabla \cdot \vec{\Pi}$. Since $\nabla \cdot A$ commutes with the Hamiltonian, as a Heisenberg picture operator it will be independent of time. Thus, if we choose our *out* states to be eigenstates of $\nabla \cdot \vec{A}(\vec{x}, +\infty)$, with eigenvalue $A_L(\vec{x})$ they will also be eigenstates of $\nabla \cdot \vec{A}(\vec{x}, -\infty)$ with the same eigenvalue.

We can always choose phases, as in the standard Schrödinger representation, so that $i \langle A', out | \vec{\Pi} = (\delta/\delta A') \langle A', out |$. Then the dependence of persistence amplitudes on the choice of A_L is determined by

$$\begin{aligned} i\delta \langle A_L, out | in \rangle &= \int d^3x \delta A_L(\vec{x}) \langle A_L, out | \left(-\frac{1}{\nabla^2} \right) \nabla \cdot \Pi(\vec{x}, +\infty) | in \rangle \\ &= \int d^3x \delta A_L(\vec{x}) \langle A_L, out | \left(+\frac{1}{\nabla^2} \right) J^0(\vec{x}, +\infty) | in \rangle. \end{aligned}$$

Since $J^0(\vec{x}, t)$ is just the infinitesimal generator of local phase changes on the charged Heisenberg fields at time t , we see that an infinitesimal change in A_L can be compensated by a local gauge transformation on the charged fields at $t = +\infty$, which determine the definition of the *out* states. The principle of gauge invariance includes the statement that state vectors differing by such gauge changes describe the same physical state, since they will give identical predictions for all gauge invariant observations. Thus we are free to fix A_L to be any convenient function. Coulomb or radiation gauge corresponds to the choice $A_L = 0$.

We have generally defined $\langle out |$ by the $t \rightarrow \infty$ limit of $\langle t | = \langle in | U(t, -\infty)$. Our choice of $\langle A_L, out |$ is implemented by the replacement $\langle in | \rightarrow \langle A_L, in |$ where the latter is the eigenstate of $\nabla \cdot A(\vec{x}, -\infty)$ with eigenvalue A_L . Then the corresponding $\langle A_L, t |$ is the corresponding eigenstate of $\nabla \cdot A(\vec{x}, t)$. Since $\langle A_L, t |$ satisfies $i(\partial/\partial t) \langle A_L, t | = \langle A_L, t | H_{COUL}(t)$, we see that replacing $\nabla \cdot A$ in H_{COUL} by A_L will yield the same $\langle A_L, out |$. In particular the radiation gauge choice $A_L = 0$ leads to the effective Hamiltonian

$$H_{eff} = \int d^3x \vec{\Pi}_T^2 + \int J^0 \left(-\frac{1}{\nabla^2} \right) J^0 + \int \left(\frac{1}{2} \partial_k \vec{A}_T \cdot \partial_k \vec{A}_T - \vec{A}_T \cdot \vec{J} \right). \quad (17.3)$$

One may use H_{eff} to compute any physical quantity. It only contains the transverse components of \vec{A} and $\vec{\Pi}$. From the canonical commutation relations for these two quantities one can easily evaluate those for the transverse components

$$[A_{Tk}(\vec{x}), \Pi_{Tm}(\vec{y})] = i \left(\delta_{km} - \frac{\nabla_k \nabla_m}{\nabla^2} \right) \delta(\vec{x} - \vec{y}).$$

The operator acting on the delta function simply reflects the fact that the l.h.s. has vanishing divergence because of the transversality of the operators.

We have obtained the effective Coulomb gauge Hamiltonian (17.3) by reduction from a gauge independent quantization procedure. A much quicker route to the same answer is to fix the gauge before quantization by setting $\nabla \cdot A = 0$ from the beginning. Then the Gauss' law constraint can be "solved" by setting

$$A^0(\vec{x}, t) = -\frac{1}{\nabla^2} J^0(\vec{x}, t) = \int d^3y \frac{J^0(\vec{y}, t)}{4\pi|\vec{x} - \vec{y}|}.$$

The longitudinal component of $\vec{\Pi}$ is also eliminated because $\nabla \cdot \vec{\Pi} = \nabla^2 A^0$. One passes to quantum mechanics by promoting only \vec{A} and Π_T to operators. (If the currents are operators, A^0 is an operator by virtue of the constraint, but it is not independent.) Then the transverse projector must appear on the r.h.s. of the canonical commutation relations.

Polarization and Helicity of Photons. An explicit realization of the commutation relations can be given in terms of creation and annihilation operators as follows:

$$A_{Tk}(\vec{x}, 0) = \int \frac{d^3 k}{\sqrt{(2\pi)^3 2|\mathbf{k}|}} [a_k(\vec{k}) e^{i\vec{k}\cdot\vec{x}} + a_k^\dagger(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}]$$

$$\Pi_{Tk}(\vec{x}, 0) = -i \int \frac{d^3 k \sqrt{|\mathbf{k}|}}{\sqrt{(2\pi)^3 2}} [a_k(\vec{k}) e^{i\vec{k}\cdot\vec{x}} - a_k^\dagger(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}]$$

with

$$[a_k(\vec{k}), a_m^\dagger(\vec{q})] = (\delta_{km} - \frac{k_k k_m}{\mathbf{k}^2}) \delta(\vec{k} - \vec{q}).$$

Inserting these into H_{eff} gives

$$H_{eff} = \int d^3 k |\vec{k}| \vec{a}^\dagger(\vec{k}) \cdot \vec{a}(\vec{k}) - \int d^3 x \vec{A}_T \cdot \vec{J} + E_0,$$

where E_0 is the usual (infinite) zero point energy of the oscillators which will be dropped from now on. This formula shows us immediately that for $\vec{J} = 0$, the quantum e.m. field is interpretable as a system of massless bosons (photons). The vacuum $|0\rangle$ is defined by $a_k(\vec{k}) |0\rangle = 0$ and the n photon state is represented by

$$a_{m_1}^\dagger(\vec{q}_1) a_{m_2}^\dagger(\vec{q}_2) \cdots a_{m_n}^\dagger(\vec{q}_n) |0\rangle.$$

Because of transversality there are two photon states for each momentum. These two polarization states will next be shown to correspond to the two helicities ± 1 of the photon.

First for fixed \vec{k} let us introduce two (in general complex) basis vectors $\vec{\epsilon}_a$, $a = 1, 2$ for the plane perpendicular to \vec{k} , satisfying $\vec{k} \cdot \vec{\epsilon}_a = 0$ and the orthonormality and completeness relations

$$\vec{\epsilon}_a \cdot \vec{\epsilon}_b^* = \delta_{ab}$$

$$\sum_a \vec{\epsilon}_a^m \cdot \vec{\epsilon}_a^{n*} = \delta_{mn} - \frac{k^m k^n}{\mathbf{k}^2}.$$

We can then introduce two independent sets of creation and annihilation operators via

$$\vec{a}(\vec{k}) = \sum_a \vec{\epsilon}_a a_a(\vec{k}).$$

We shall relate the multiplicity associated with the index a to the spin of the photon. First recall the classical expression for the angular momentum carried by the e.m. field,

$$\vec{J} = \int d^3 x \vec{x} \times (\vec{E} \times \vec{B})$$

$$= \int d^3 x \sum_k E_k(\vec{x} \times \vec{\nabla}) A_k - \int d^3 x \vec{x} \times (\vec{E} \cdot \vec{\nabla}) \vec{A}.$$

We can recognize the first term in the last line as the “orbital” angular momentum, which will not contribute to the helicity of a one photon state. This is because acting on a one photon state the ∇ is replaced by \vec{k}

and because of the cross product the term will be perpendicular to \vec{k} . The second term, after an integration by parts becomes

$$\begin{aligned}\vec{S} &= \int d^3x \vec{E} \times \vec{A} \\ &= -i \int d^3k \vec{a}^\dagger(\vec{k}) \times \vec{a}(\vec{k}).\end{aligned}$$

Applying \vec{S} to a one photon state $a_a^\dagger(\vec{k})|0\rangle$, yields

$$\vec{S} a_a^\dagger(\vec{k})|0\rangle = i \sum_b (\vec{\epsilon}_a \times \vec{\epsilon}_b^*) a_b^\dagger(\vec{k})|0\rangle.$$

Thus we see that the 2×2 matrix $\vec{S}_{ab} = i\vec{\epsilon}_a \times \vec{\epsilon}_b^*$ acts as a spin matrix on the index of the creation operator. To get the helicity interpretation, consider the case of $\vec{k} = k\hat{z}$. Then the helicity matrix is

$$S_{ab}^3 = i(\epsilon_a^1 \epsilon_b^{2*} - \epsilon_a^2 \epsilon_b^{1*}).$$

This matrix is $diag\{1, -1\}$ with the choices

$$\vec{\epsilon}_1 = (1, i, 0)/\sqrt{2} \quad \vec{\epsilon}_2 = (1, -i, 0)/\sqrt{2},$$

so with this choice of polarization vectors, a_1^\dagger creates a photon with helicity +1 and a_2^\dagger creates a photon with helicity -1. This establishes that the photon is a spin one particle. There is no zero helicity state for the photon: this is consistent with Poincaré invariance because the photon is massless.

The polarization vector enters scattering amplitudes multilinearly, with a factor of ϵ for each incoming photon and a factor ϵ^* for each outgoing photon. Its four-vector index forms a Minkowski scalar product with that of the vertex coupling the gauge potential to the charged fields. According to gauge invariance, this vertex satisfies current conservation: its scalar product with the momentum entering it gives zero. Thus changing each polarization vector by an amount proportional to its four-momentum leaves the scattering amplitude unaltered. In Coulomb gauge the polarization vector is of the form $\epsilon = (\boldsymbol{\epsilon}, 0)$ with $\mathbf{k} \cdot \boldsymbol{\epsilon} = 0$, so $k_\mu \epsilon^\mu = 0$. But $k_\mu k^\mu = 0$ since the photon is massless. Thus we can characterize the polarization vector completely by the covariant condition $k_\mu \epsilon^\mu = 0$. Any further specification, *e.g.* $\epsilon^0 = 0$, is merely a gauge choice which can be made at will and exploited to simplify detailed calculations. This is particularly advantageous in the calculation of Compton scattering for polarized photons.

17.2. CHARGED FIELDS INTERACTING WITH THE QUANTIZED ELECTROMAGNETIC FIELD

We have seen that when the e.m. field interacts with a conserved current, the time component of the current enters the Hamiltonian only through the coefficient in the term linear in A_0 so that the coefficient of A_0 is just the Gauss' Law constraint with nonzero charge density. This feature is quite general, even when the e.m. field couples to dynamical charged fields. This is obvious in the case of the Dirac field because the Dirac field Hamiltonian is linear in A_μ . The A dependence of the scalar field Hamiltonian includes quadratic pieces in the *spatial* components of the potential, but nonetheless A_0 still enters only linearly after the Hamiltonian is expressed solely in terms of coordinates and momenta. This means that the elimination of A_0 in the passage to Coulomb gauge proceeds exactly as in the previous section.

Thus, the Hamiltonian for e.m. field plus charged fields in Coulomb gauge will quite generally be of the form

$$H_{eff} = \int d^3x \left(\frac{1}{2} \mathbf{\Pi}_T^2 + \frac{1}{2} \partial_k \mathbf{A}_T \cdot \partial_k \mathbf{A}_T - \mathbf{A}_T \cdot \mathbf{J}_e \right) + \int (J_e^0 + j^0) \left(-\frac{1}{2\nabla^2} \right) (J_e^0 + j^0) + H_{fields}|_{A_0=0}, \quad (17.4)$$

where J_e^μ is an optional external current, j^μ is the current operator for the fields, and the subscript *fields* refers to the Hamiltonian operator for any dynamical fields in the system. For the Dirac field

$$j^0 = Q[\bar{\psi}, \gamma^0 \psi] \\ H_{fields} = \int d^3x \bar{\psi} \left(\frac{1}{i} \boldsymbol{\gamma} \cdot \boldsymbol{\partial} + m - Q \boldsymbol{\gamma} \cdot \mathbf{A} \right) \psi,$$

and for the scalar field,

$$j^0 = -iQ(\pi\phi - \phi^\dagger\pi^\dagger) \\ H_{fields} = \int d^3x (\pi\pi^\dagger + m^2\phi^\dagger\phi + (\nabla + iQ\mathbf{A})\phi^\dagger \cdot (\nabla - iQ\mathbf{A})\phi).$$

One obstacle to formulating an efficient perturbation theory for systems with interacting quantum fields is that one can't "turn off" the interactions at early and late times as is possible with externally applied fields. Thus *out* and *in* states are eigenstates of complicated interacting Hamiltonians. One approach to this difficulty is to artificially make the coupling constants time dependent and force them to vanish at early and late times. Another approach, which we shall favor, is to relax the requirement that the initial and final states be eigenstates of the Hamiltonian with vanishing external fields. Then one calculates in first instance a quantity that is not of immediate interest, but which can be simply related to such quantities.

A quantity of more or less direct physical interest is the vacuum expectation value of the time ordered product of several quantum fields. More generally the *outin* matrix element of such a time ordered product is relevant if time varying external fields are present. So let us consider how to obtain this quantity in perturbation theory by first calculating with general initial and final states. Using the evolution operator and assuming $t_1 > t_2 > \dots > t_n$, we therefore consider

$$\langle f | U(\infty, -\infty) T[\Omega_1(t_1) \dots \Omega_n(t_n)] | i \rangle = \langle f | U(\infty, t_1) \Omega_{S1} U(t_1, t_2) \dots U(t_{n-1}, t_n) \Omega_{Sn} U(t_n, -\infty) | i \rangle.$$

Choose the time T so that all external fields vanish for times earlier than $-T$ and later than T . Then

$$U(t_n, -\infty) | i \rangle = U(t_n, -T) e^{-i(\infty-T)H_S} | i \rangle \\ = U(t_n, -T) e^{-i(\infty-T)E_G} \sum_r e^{-i(\infty-T)(E_r - E_G)} | r \rangle \langle r | i \rangle.$$

We would now like to argue that the infinite oscillations wash out all contributions but the (assumed nondegenerate^{*}) ground state. In a field theory this is quite plausible since the excited states correspond to particles so the sum over r is really an integral over a range of continuous energies. But even without

* There are interesting cases of degenerate vacua, when there is "spontaneous symmetry breakdown." In such cases the choice of initial and final states determines which of the degenerate vacua is picked out.

this smearing, we can make the washing out rigorous by calculating with imaginary time: $it = \beta > 0$. Then $i\infty$ is really $+\infty$ and all excited states are damped exponentially. Massless particle states could introduce a subtlety here, but the part of phase space that is not exponentially damped is infinitesimal: this has the effect of changing exponential damping to a power law damping. If we buy this argument, then we can assert quite generally that $U(t_n, -\infty) |i\rangle = U(t_n, -\infty) |0\rangle \langle 0|i\rangle$ and similarly $\langle f|U(\infty, t_1) = \langle f|0\rangle \langle 0|U(\infty, t_1)$.

Since we take (as usual) the Heisenberg and Schrödinger pictures to coincide at $t = -\infty$, then $|in\rangle = |0\rangle$ and $\langle out| = \langle 0|U(\infty, -\infty)$. Thus we have obtained the relation

$$\begin{aligned} \langle f|U(\infty, -\infty)T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|i\rangle = \\ \langle f|0\rangle \langle 0|i\rangle \langle out|T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|in\rangle. \end{aligned}$$

In other words calculating with any initial and final states that have finite overlap[†] with the true ground state gives us a constant times the desired matrix element. We can easily evaluate the multiplicative constant by considering by the same reasoning

$$\begin{aligned} \langle f|U(\infty, -\infty)|i\rangle &= \langle f|0\rangle \langle 0|i\rangle \langle out|in\rangle \\ &\rightarrow e^{-2i\infty E_G} \langle f|0\rangle \langle 0|i\rangle \quad \text{External Fields} = 0. \end{aligned}$$

Putting this into our relation we obtain

$$\begin{aligned} \langle out|T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|in\rangle = \\ e^{-2i\infty E_G} \frac{\langle f|U(\infty, -\infty)T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|i\rangle}{\langle f|U(\infty, -\infty)_{Ext=0}|i\rangle}, \end{aligned} \quad (4.13)$$

where the subscript on U in the denominator denotes vanishing external fields. In field theory applications E_G is the energy of the vacuum, which is zero if we measure all energies relative to that of the vacuum. In the absence of gravity all physical quantities depend only on energy differences, so we lose nothing by doing this. Gravity couples directly to the energy density and therefore is sensitive to the energy as opposed to energy differences, but then E_G only appears in the combination $\Lambda \equiv E_G + \Lambda_0$, with Λ_0 the “bare” cosmological constant cosmological constant. Replacing Λ_0 by Λ in effect sets $E_G = 0$.

The formula (4.13) is a convenient starting point for developing perturbation theory. Any breakup

$$H_S(t) = H_0(t) + H'(t)$$

determines an interaction picture defined by

$$\Omega_I(t) = U_0^{-1}(t, -\infty)\Omega_S U_0(t, -\infty) = U_I^{-1}(t, -\infty)\Omega(t)U_I(t, -\infty),$$

where

$$i\dot{U} = H_S(t)U = UH(t)$$

$$i\dot{U}_0 = U_0 H_0(t)$$

$$i\dot{U}_I = H_I(t)U_I$$

[†] The infinite number of degrees of freedom in quantum field theory requires care here: the overlap between different states in a theory with n degrees of freedom is typically f^n with $f < 1$. Since $n = \infty$, we should expect $\langle f|0\rangle \langle 0|i\rangle \sim e^{-\infty}$. In field theory $n = \infty$ because the volume of space is infinite and because space is continuous. Thus strict application of the above relation should be done in the presence of both an infrared and ultraviolet cutoff, which can then be removed after extracting the desired amplitude.

and all U 's are the identity at $t = -\infty$. Then the evolution operator satisfies

$$U(t_1, t_2) = U_0(t_1, -\infty)U_I(t_1, t_2)U_0(t_2, -\infty).$$

Plugging these relations into (4.13) then gives

$$\begin{aligned} \langle out|T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|in\rangle = \\ e^{-2i\infty E_G} \frac{\langle f|U_0(\infty, -\infty)T[U_I(\infty, -\infty)\Omega_{1I}(t_1)\cdots\Omega_{nI}(t_n)]|i\rangle}{\langle f|U_0(\infty, -\infty)_{Ext=0}U_I(\infty, -\infty)_{Ext=0}|i\rangle}. \end{aligned} \quad (4.14)$$

This formula is completely general: we have even allowed H_0 to contain time varying external fields, which is hardly ever done in practice. Since all operators in this formula are in interaction picture, it is most convenient to choose $|i\rangle, |f\rangle$ to have simple properties with respect to $H_{0I}(-\infty)$. Let us call the ground state of this operator $|in, 0\rangle$. Then $\langle in, 0|U_0(\infty, -\infty)$ is the ground state of $H_{0I}(+\infty)$ and therefore deserves the name $\langle out, 0|$. When all external fields vanish, H_{0I} is time independent and we call its ground state $|0, I\rangle \equiv |in, 0\rangle$ and its ground state energy E_0 . Then $\langle in, 0|U_0(\infty, -\infty)_{Ext=0} = e^{-2i\infty E_0} \langle 0, I|$. Thus choosing $|i\rangle = |f\rangle = |0, I\rangle = |in, 0\rangle$ we obtain the useful formula

$$\begin{aligned} \langle out|T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|in\rangle = \\ e^{-2i\infty(E_G - E_0)} \frac{\langle out, 0|T[U_I(\infty, -\infty)\Omega_{1I}(t_1)\cdots\Omega_{nI}(t_n)]|in, 0\rangle}{\langle 0, I|U_I(\infty, -\infty)_{Ext=0}|0, I\rangle}. \end{aligned} \quad (4.15)$$

In the usual case where we do not include external fields in H_0 , the formula simplifies further

$$\begin{aligned} \langle out|T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|in\rangle = \\ e^{-2i\infty E_G} \frac{\langle 0, I|T[U_I(\infty, -\infty)\Omega_{1I}(t_1)\cdots\Omega_{nI}(t_n)]|0, I\rangle}{\langle 0, I|U_I(\infty, -\infty)_{Ext=0}|0, I\rangle}. \end{aligned} \quad (4.16)$$

Using the Wick expansion one can describe the perturbation series for the numerators and denominators of these formulas using Feynman diagrams. The diagrams contributing to the denominator are all those completely disconnected from either external fields or from the points assigned to the operators in the numerator. The numerator contains this same sum of diagrams as a multiplicative factor. Thus the division by the denominator is achieved by simply deleting all such disconnected ‘‘vacuum’’ diagrams from the expansion of the numerator.

18. Vacuum Polarization

Before leaving external field problems, it is interesting to consider the effect an external field has on the vacuum (negative energy sea in the Dirac case). We shall calculate the response of the vacuum to the application of a weak electromagnetic field. For definiteness consider the Dirac field described by

$$H_A = H_0 - \int d^3x j^\mu(\mathbf{x}, t) A_\mu(\mathbf{x}, t).$$

Assume that $A \rightarrow 0$ as $t \rightarrow -\infty$ and that the system starts out in the ground state of $H_A(-\infty) = H_0(-\infty)$.

We should expect the field to induce charge and current densities in the vacuum. A simple measure of these induced currents is the expectation value of the Heisenberg picture current operator in the system state $\langle in | j^\mu(x) | in \rangle_A$. We may express this matrix element in interaction picture and then develop it in an expansion in powers of A . In the limit of very weak fields we can neglect all terms beyond those linear in A :

$$\begin{aligned} \langle in | j^\mu(x) | in \rangle &= \langle in | U_I^{-1}(t, -\infty) j_I^\mu(x) U_I(t, -\infty) | in \rangle_A \\ &= \langle 0, I | \left(T e^{i \int_{-\infty}^t j_I \cdot A} \right)^\dagger j_I^\mu(x) T e^{i \int_{-\infty}^t j_I \cdot A} | 0, I \rangle \\ &\approx i \int d^4y \theta(t - t_y) \langle 0, I | [j_I^\mu(\mathbf{x}, t), j_I^\nu(\mathbf{y})] | 0, I \rangle A_\nu(y). \end{aligned}$$

There is no term independent of A because the vacuum expectation value of the current vanishes in the absence of applied fields.

This is the linear response to an applied field and is characterized by the response function

$$R^{\mu\nu}(x) \equiv i\theta(t) \langle 0 | [j^\mu(x), j^\nu(0)] | 0 \rangle$$

where here and in the following we drop the subscripts I and it is understood that the currents are those of free fields. An important physical property of the response function is Einstein causality: $R^{\mu\nu}$ vanishes for spacelike argument $x^2 > 0$ as follows from the fact that local operators commute at space-like separations. Thus application of an external field at the origin at $t = 0$ cannot evoke a response at \mathbf{x} until enough time has elapsed for light to travel from the origin to \mathbf{x} . This property is *not* shared by the time ordered product.

Retarded Commutators from Time Ordered Products The Wick expansion we have developed works best for time ordered products, so it is helpful that we can work out a relationship between the response function and the expectation value of the time-ordered product. This relationship is a general one that depends only on the time variable, so we suppress spatial and internal labels and consider two hermitian operators $O_1(t), O_2(t)$. We shall actually relate the Fourier transforms of the two quantities:

$$\begin{aligned} R(\omega) &\equiv i \int dt e^{i\omega t} \theta(t) \langle G | [O_1(t), O_2(0)] | G \rangle \\ T(\omega) &\equiv i \int dt e^{i\omega t} \langle G | T[O_1(t) O_2(0)] | G \rangle \end{aligned}$$

where $|G\rangle$ is the Ground State of the system, *assumed* to be nondegenerate. Now using $\theta(t) = 1 - \theta(-t)$ we

have

$$\theta(t)[O_1(t), O_2(0)] = T[O_1(t)O_2(0)] - O_2(0)O_1(t)$$

so the difference between R and T involves

$$\begin{aligned} & -i \int dt e^{i\omega t} \langle G | O_2(0) O_1(t) | G \rangle \\ &= -i \int dt e^{i\omega t} \sum_n \langle G | O_2(0) | n \rangle \langle n | O_1(0) | G \rangle e^{-i(E_G - E_n)t} \\ &= -2\pi i \delta(\omega) \langle G | O_2(0) | G \rangle \langle G | O_1(0) | G \rangle \\ & -i \int dt e^{i\omega t} \sum_{n \neq G} \langle G | O_2(0) | n \rangle \langle n | O_1(0) | G \rangle e^{-i(E_G - E_n)t} \end{aligned}$$

The important feature of this result is that by virtue of the fact that E_G is the lowest energy eigenvalue, the r.h.s. *vanishes* for positive frequency $\omega > 0$. Thus in this case $R(\omega) = T(\omega)$. Next we find a relation for negative frequency. For this case we relate R to the anti-time-ordered product:

$$\theta(t)[O_1(t), O_2(0)] = O_1(t)O_2(0) - \bar{T}[O_1(t)O_2(0)].$$

Now inserting a complete set of states allows us to conclude that the Fourier transform of the first term vanishes for negative frequency $\omega < 0$. Thus in this case we have $R(\omega) = -\bar{T}(\omega)$ where

$$\bar{T}(\omega) \equiv i \int dt e^{i\omega t} \langle G | \bar{T}[O_1(t)O_2(0)] | G \rangle = -T^*(-\omega).$$

where we used the assumption that O_1, O_2 are hermitian.

In summary we have found that

$$R(\omega) = \begin{cases} T(\omega) & \omega > 0 \\ T^*(-\omega) & \omega < 0. \end{cases}$$

For ω near zero we observe that the ground state contributes to T but not to R . The contribution to T is

$$T_G(\omega) = 2\pi i \delta(\omega) \langle G | O_2(0) | G \rangle \langle G | O_1(0) | G \rangle.$$

If there is a gap separating E_G from the rest of the spectrum this is the only zero frequency discrepancy between R and T .

Finally when we consider this relationship for field operators, it is natural to quote it for the spatial and temporal Fourier transform:

$$R(p^0, \mathbf{p}) = \begin{cases} T(p^0, \mathbf{p}) & p^0 > 0 \\ T^*(-p^0, -\mathbf{p}) & p^0 < 0. \end{cases}$$

Again the vacuum contributes to T (and not R) the amount

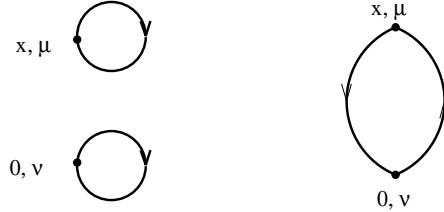
$$T_G(p^0) = (2\pi)^4 i \delta^4(p) \langle 0 | O_1 | 0 \rangle \langle 0 | O_2 | 0 \rangle.$$

Calculation of Vacuum Polarization If we Fourier transform the current induced by an external field, we obtain

$$\begin{aligned}\tilde{j}^\mu(k) &= \int d^4x e^{-ik \cdot x} \langle in | j^\mu(x) | in \rangle \\ &= \int d^4x e^{-ik \cdot x} \int d^4y R^{\mu\nu}(x-y) A_\nu(y) + O(A^2) \\ &= R^{\mu\nu}(k) \tilde{A}_\nu(k) + O(A^2)\end{aligned}$$

and we have just obtained the relation of $R^{\mu\nu}$ to $T^{\mu\nu}$.

To calculate $T^{\mu\nu}$ we first apply the Wick expansion to the time ordered product of four Dirac fields contained in the two current amplitude. There are two distinct contractions corresponding to the diagrams



The disconnected diagrams vanish because $\langle 0 | j^\mu | 0 \rangle = 0$ in the absence of external fields. (This is a simple consequence of charge conjugation invariance.) The unique connected diagram has the value

$$-Q^2 \text{Tr}[\gamma^\mu S_F(x) \gamma^\nu S_F(-x)]$$

where the minus sign comes from the single closed fermi loop. Inserting the known Fourier representation for S_F and carrying out the integration over x in the evaluation of $T^{\mu\nu}$ leads to

$$T^{\mu\nu}(k) = iQ^2 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left(\gamma^\mu \frac{m - p \cdot \gamma}{m^2 + p^2 - i\epsilon} \gamma^\nu \frac{m - (p-k) \cdot \gamma}{m^2 + (p-k)^2 - i\epsilon} \right).$$

We immediately see from this expression that the integration over momentum is quadratically divergent at high momentum. The origin of this divergence is that $S_F(x)$ behaves like $1/x^3$ at small x which means that the two current amplitude behaves like $1/x^6$ which means that its Fourier transform is ill-defined.

Before dealing with this divergence, let us simplify the integrand by first evaluating the trace

$$\begin{aligned}N^{\mu\nu}(p, k) &\equiv \text{Tr}(\gamma^\mu (m - p \cdot \gamma) \gamma^\nu (m - (p-k) \cdot \gamma)) \\ &= 8p^\mu p^\nu - 4(p^\mu k^\nu + p^\nu k^\mu) - 4\eta^{\mu\nu} (m^2 + p \cdot (p-k))\end{aligned}$$

and secondly combining denominators using the Feynman trick

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax + B(1-x)]^2}$$

which is trivial to derive. Then

$$T^{\mu\nu}(k) = iQ^2 \int \frac{d^4p}{(2\pi)^4} \int_0^1 dx \frac{N^{\mu\nu}(p, k)}{[m^2 + (p-kx)^2 - i\epsilon + x(1-x)k^2]^2}$$

where we have completed the square in the denominator.

Next we do a step which is not quite legitimate in view of the quadratic divergence, which is to change integration variables $p \rightarrow p + kx$. After this the denominator depends only on p^2 , so all terms in the numerator linear in p^μ integrate to zero and can be dropped. Furthermore terms of the form $p^\mu p^\nu$ can be replaced by $p^2 \eta^{\mu\nu}/4$, since $\int d^4 p f(p^2) p^\mu p^\nu$ must be proportional to $\eta^{\mu\nu}$ and the proportionality constant is then determined to be $\int d^4 p f(p^2) p^2/4$ by comparing the trace of both sides. Thus we have the replacements

$$\begin{aligned} N^{\mu\nu}(p + xk, k) &\rightarrow 8\left(\frac{p^2}{4}\eta^{\mu\nu} + x^2 k^\mu k^\nu\right) - 8xk^\mu k^\nu - 4\eta^{\mu\nu}(m^2 + p^2 - x(1-x)k^2) \\ &\rightarrow -2\eta^{\mu\nu}p^2 + 4x(1-x)(k^2\eta^{\mu\nu} - 2k^\mu k^\nu) - 4m^2\eta^{\mu\nu} \end{aligned}$$

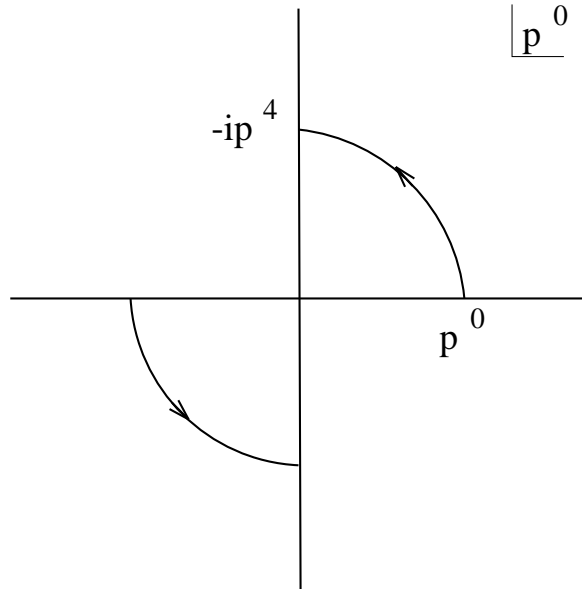
After all these steps so far we have reduced the integrals to

$$T^{\mu\nu}(k) = iQ^2 \int \frac{d^4 p}{(2\pi)^4} \int_0^1 dx \frac{-2\eta^{\mu\nu}p^2 + 4x(1-x)(k^2\eta^{\mu\nu} - 2k^\mu k^\nu) - 4m^2\eta^{\mu\nu}}{[m^2 + p^2 - i\epsilon + x(1-x)k^2]^2}.$$

Next we wish to evaluate the p integral. It is easiest to think about this evaluation after the Wick rotation to Euclidean momenta, so the integral is over 4 dimensional Euclidean space and the integrand is $O(4)$ invariant. Then the angular integrals can be done and the integral reduced to a one dimensional one. The rotation of the p^0 contour to the imaginary axis must avoid the singularities due to the vanishing of the denominator which occurs at

$$p^0 = \pm \sqrt{m^2 + \mathbf{p}^2 + x(1-x)k^2 - i\epsilon}$$

These poles remain in the fourth and second quadrant of the complex p^0 plane for all values of k^2 . However they get infinitesimally close to the imaginary axis for $x(1-x)k^2 \leq -m^2 - \mathbf{p}^2$ which we shall see is responsible for singular behavior in the result as a function of k^2 . As long as we stick to $k^2 > -4m^2$, though, the poles stay well within their respective quadrants, and a counterclockwise contour rotation by 90 degrees encounters no singularities.



After the Wick rotation we change variables to $p^0 = ip^4$ so that $d^4 p = id^4 p_E$ and $p^2 = \mathbf{p}^2 + (p^4)^2$. Going to polar coordinates, $d^4 p_E = p^3 dp d\Omega$, we wish to evaluate the angular integrals $d\Omega$. A useful trick to do this in any number of dimensions is to integrate a Gaussian $e^{-\vec{p}^2}$ in both Cartesian and polar coordinates. In

Cartesian coordinates in D dimensions we get $\pi^{D/2}$. In polar coordinates it is

$$\Omega_D \int_0^\infty p^{D-1} dp e^{-p^2} = \Omega_D \frac{1}{2} \Gamma(D/2)$$

Comparing we arrive at

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}.$$

Notice that for $D = 1, 2, 3$ this gives the well known results $2, 2\pi, 4\pi$. For our case $D = 4$ and the result is $2\pi^2$. We also record here the useful formula

$$\int d^D p \frac{(p^2)^m}{(p^2 + A^2)^n} = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty dp \frac{p^{D-1+2m}}{(p^2 + A^2)^n} = \frac{A^{D+2m-2n} \pi^{D/2} \Gamma(m + D/2) \Gamma(n - m - D/2)}{\Gamma(D/2) \Gamma(n)}.$$

The fact that the r.h.s. is a perfectly defined analytic function of D, m, n allows for dimensional regularization as we shall see later.

Putting all this together and cutting off the p integral at Λ we have so far

$$T_\Lambda^{\mu\nu}(k) = -\frac{Q^2}{8\pi^2} \int_0^\Lambda p^3 dp \int_0^1 dx \frac{-2\eta^{\mu\nu}(p^2 + 2m^2) + 4x(1-x)(k^2 \eta^{\mu\nu} - 2k^\mu k^\nu)}{[m^2 + p^2 - i\epsilon + x(1-x)k^2]^2}.$$

The p integrals are now elementary:

$$\begin{aligned} \int_0^\Lambda \frac{p^3 dp}{[p^2 + C]^2} &= \frac{1}{2} \left[\ln \frac{\Lambda^2 + C}{C} - 1 \right] + \frac{1}{2} \frac{C}{\Lambda^2 + C} \\ &= \frac{1}{2} \left[\ln \frac{\Lambda^2}{C} - 1 \right] + O(\Lambda^{-2}) \\ \int_0^\Lambda \frac{p^5 dp}{[p^2 + C]^2} &= \Lambda^2 - C \ln \frac{\Lambda^2 + C}{C} - \frac{1}{2} \frac{\Lambda^4}{\Lambda^2 + C} \\ &= \frac{1}{2} \Lambda^2 + \frac{1}{2} C - C \ln \frac{\Lambda^2}{C} + O(\Lambda^{-2}), \end{aligned}$$

where for us $C = m^2 + x(1-x)k^2 - i\epsilon$. Putting these results into the expression for T gives

$$\begin{aligned} T_\Lambda^{\mu\nu}(k) &= -\frac{Q^2}{8\pi^2} \int_0^1 dx \left[-\eta^{\mu\nu}(\Lambda^2 - C - 2C \ln \frac{\Lambda^2}{C} - 1) \right. \\ &\quad \left. - (2\eta^{\mu\nu} m^2 - 2x(1-x)(k^2 \eta^{\mu\nu} - 2k^\mu k^\nu) \left[\ln \frac{\Lambda^2}{C} - 1 \right] \right] \\ &= -\frac{Q^2}{2\pi^2} \int_0^1 dx x(1-x)(k^2 \eta^{\mu\nu} - k^\mu k^\nu) \left(\ln \frac{\Lambda^2}{m^2 + x(1-x)k^2 - i\epsilon} - 1 \right) \\ &\quad + \frac{Q^2}{8\pi^2} \eta^{\mu\nu} (\Lambda^2 - m^2 - \frac{k^2}{6}). \end{aligned} \tag{18.1}$$

We could do the last integral over x , but it is actually easier to see the properties of T directly from the integral representation (18.1).

We have been casual about the way we cutoff the momentum integral, and now we can see a bad consequence of this: a violation of gauge invariance. In Fourier components a gauge transformation on the external field has the form $\tilde{A}_\mu(k) \rightarrow \tilde{A}_\mu + k_\mu \tilde{\Lambda}(k)$. The induced current was given by $R^{\mu\nu}(k)\tilde{A}_\nu(k)$, so gauge invariance would imply $R^{\mu\nu}(k)k_\nu = 0$, and in view of the relation between R and T , $T^{\mu\nu}(k)k_\nu = 0$. Clearly, the last line of (18.1) fails to satisfy this condition. The reason for this error can be traced to insufficient care with the manner in which we regularized the divergent integral. It is fortunate that the momentum dependence of the erroneous terms is a simple polynomial. This is in fact a characteristic of all such errors induced by ultraviolet divergences: Differentiating the integral a finite number of times with respect to the external momenta renders it convergent, so a finite number of derivatives must kill the mistake. In this case it would require three derivatives to kill the mistake. We shall later discuss gauge invariant regularization procedures that prevent such mistakes from occurring, provided of course that the theory can be consistently quantized. But for now we shall be satisfied with simply adjusting the polynomial dependence of our results to be consistent with gauge invariance. The nonpolynomial part of T can of course not be removed by such an adjustment. Making this adjustment, and at the same time absorbing the -1 in a rescaling of the cutoff, we then obtain

$$T_{GI}^{\mu\nu}(k) = (k^\mu k^\nu - k^2 \eta^{\mu\nu})T(k^2)$$

$$T(k^2) = \frac{Q^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \frac{\Lambda^2 e^{-\gamma}}{m^2 + x(1-x)k^2 - i\epsilon}.$$

It is important to appreciate that gauge invariant regularization does not cure the problem of ultraviolet divergences, although it does reduce its severity. Our initial expression for T was quadratically divergent, but we have seen that gauge invariance effectively reduces the divergence to a logarithmic one. We shall see that this last divergence, although present in the quantities we are calculating, disappears after expressing the answer in terms of physically measurable parameters. Note also that our polynomial adjustment of $T^{\mu\nu}$ to make it gauge invariant allows an undetermined constant γ . However this ambiguity is linked to the cutoff dependence, and will disappear along with the latter in physical quantities.

The Physics of Vacuum Polarization Our result for $T^{\mu\nu}$ can now be used to give us the response function

$$R_{GI}^{\mu\nu}(k) = (k^\mu k^\nu - k^2 \eta^{\mu\nu})R(k^2)$$

where

$$R(k^2) = \begin{cases} T(k^2) & k^0 > 0 \\ T^*(k^2) & k^0 < 0 \end{cases}.$$

Note that since T is real for $k^2 > -4m^2$, the two cases merge for that range of momentum. Since we have incorporated gauge invariance in our answer, we are free to fix a convenient gauge for discussing the physical interpretation of our result. Let us choose Lorentz gauge, $k^\mu \tilde{A}_\mu = 0$. Then the Fourier transform of the current induced by the external field is simply

$$\langle j_\mu(k) \rangle = -k^2 R(k^2) \tilde{A}_\mu^e(k).$$

We must now recognize that the induced currents will produce induced fields via Maxwell's equations. As long as A^e the external field is sufficiently weak the induced currents and the induced field A^{IND} will also be weak, and it will be consistent to assert that the total current is given by the response function times the

total field $A^e + A^{IND}$.

$$\langle j_\mu(k) \rangle_{TOTAL} = -k^2 R(k^2) (\tilde{A}_\mu^e(k) + \tilde{A}_\mu^{IND}(k)).$$

It is this total current that we should use in Maxwell's equations to calculate A^{IND}

$$k^2 \tilde{A}_\mu^{IND}(k) = \langle j_\mu(k) \rangle_{TOTAL} = -k^2 R(k^2) (\tilde{A}_\mu^e(k) + \tilde{A}_\mu^{IND}(k)).$$

This gives a self-consistent equation for A^{IND} in terms of the external field.

$$\tilde{A}_\mu^{IND}(k) = -\frac{R(k^2)}{1 + R(k^2)} \tilde{A}_\mu^e(k).$$

Finally, if we add the induced field to the external field, we obtain the total field

$$\tilde{A}_\mu^{TOT}(k) = \frac{1}{1 + R(k^2)} \tilde{A}_\mu^e(k).$$

In summary we have calculated the total electromagnetic field that arises in the “medium” of the Dirac sea in the presence of an externally applied field. The externally applied fields are what are traditionally called the \vec{D} and \vec{H} fields. (Recall that the sources of these fields are the external charge and current density respectively.) On the other hand the total fields are traditionally given the name \vec{E} and \vec{B} . The dielectric “constant” of the medium is defined by $\vec{D} = \epsilon \vec{E}$ and the magnetic permeability by $\vec{B} = \mu \vec{H}$. Thus we can interpret our calculation by attributing a k dependent dielectricity and magnetic permeability to the Dirac sea

$$\epsilon(k^2) = 1/\mu(k^2) = 1 + R(k^2).$$

Since R is positive for static fields ($k^0 = 0$), the vacuum is a polarizable diamagnetic medium. The fact that $\epsilon = 1/\mu$ means that the velocity of light is unaltered by the medium (*i.e.* the medium preserves Poincaré invariance). The effectiveness of the medium in screening external fields is reduced at shorter wavelength (R decreases as k^2 increases).

Charge Renormalization We now come to the resolution of the logarithmic divergence that remains in our expression for $R(k^2)$. We begin by asking how we measure charge. We seem to have particles and antiparticles of charge $\pm Q$ in the theory. However this is not the measured charge, the charge we could define by $e_{ph}^2 \equiv \lim_{R \rightarrow \infty} 4\pi R^2 F(R)$ where R is the spatial separation between two such charged particles and F is the force exerted by one on the other. In other words, Q represents the external or “bare” charge, which acts as source to the \vec{D} field. The Fourier component of the latter field is just $-iQ\mathbf{k}/\mathbf{k}^2$. The measured force is given by $Q\vec{E}$ the Fourier component of which is $-iQ^2\mathbf{k}/[\mathbf{k}^2(1 + R(k^2))]$. The long distance part of the force is controlled by the Fourier components with $k \approx 0$ which are clearly those of a Coulomb force with effective charge squared of

$$e^2 = \frac{Q^2}{1 + R(0)}. \quad (18.2)$$

Putting $k = 0$ in our expression for R gives us

$$e^2 = \frac{Q^2}{1 + (Q^2/12\pi^2) \ln(\Lambda^2/m^2)}.$$

It is e and not the parameter Q that we measure in experiments, all of which are performed within the “medium” represented by the vacuum. The fine structure constant is $\alpha = e^2/4\pi \approx 1/137$.

If e is the measured charge, then the measured electric field should be related to measured force by

$$\begin{aligned}\vec{F} = e\vec{E}_{meas} &= Q\vec{E} = \frac{Q}{1 + R(k^2)}\vec{D} \\ &= \frac{e(1 + R(0))}{1 + R(0) + [R(k^2) - R(0)]} \frac{\vec{D}}{\sqrt{1 + R(0)}} \\ &= \frac{e}{1 + [R(k^2) - R(0)]/(1 + R(0))} \frac{\vec{D}}{\sqrt{1 + R(0)}}\end{aligned}$$

Now since \vec{D} is simply proportional to Q , $\vec{D}_{meas} \equiv \vec{D}/\sqrt{1 + R(0)}$ is what we can call the measured \vec{D} field since it has Q replaced by e . Thus we have the following relationship between measured fields

$$\vec{E}_{meas} = \frac{1}{1 + [R(k^2) - R(0)]/(1 + R(0))} \vec{D}_{meas}.$$

The measured dielectric constant is accordingly

$$\epsilon(k^2) = 1 + \frac{R(k^2) - R(0)}{1 + R(0)} \approx 1 + \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \frac{m^2}{m^2 + x(1-x)k^2}$$

and we see that the cutoff dependence has disappeared when we express measured quantities in terms of measured parameters. This is what is known as Renormalizability, and is a feature of a wide class of quantum field theories. Such theories encounter infinities in intermediate stages of a given calculation, but the measurable quantities always come out finite.

After renormalization, the dielectric constant is fixed to be 1 at $k = 0$, *i.e.* the medium is effectively absent then. Before, we argued that the effects of the medium should be reduced at large k . By going to large k we should begin to see more and more of the bare charge. We can phrase this by defining a k dependent coupling by $\alpha(k^2) = e^2/4\pi\epsilon(k^2)$. Then as k^2 increases from zero, corresponding to shorter distances, α increases until it blows up at some finite value of k^2 . It's clear that this will happen because the \ln starts out at zero goes negative and behaves monotonically without bound. The pole occurs at a value of $k^2 \sim m^2 \exp(12\pi^2/e^2)$. We have seen how poles in amplitudes are associated with particles. Unfortunately this pole occurs at space-like momentum, *i.e.* imaginary mass, a tachyon. This is the physically unacceptable Landau Ghost. If it were really present it would signify an inconsistency of electrodynamics. Fortunately, the whole issue is completely open since we have made approximations in our calculation that amount to weak coupling perturbation theory, and as we have seen the effective coupling gets strong at values of k^2 much less than the ghost mass, thus invalidating perturbation theory.

One way to understand why this approximation has led to this problem is to return to the bare expressions

$$e^2 = \frac{1}{1/Q^2 + (1/12\pi^2) \ln(\Lambda^2/m^2)}.$$

Now in the renormalization procedure, we attempt to take $\Lambda \rightarrow \infty$ holding e fixed. But this is only possible if Q is imaginary, which would mean we started with a Hamiltonian which was not hermitian. In order to escape this conclusion, the relation between bare and measured coupling would have to be fundamentally altered by higher order corrections. This is a logical possibility, but many field theorists including Landau doubted that the problem would go away. At this point it is appropriate to mention that for some quantum

field theories the sign in front of the logarithmic divergence is opposite to that in QED:

$$g^2 = \frac{1}{1/g_0^2 - b \ln(\Lambda^2/m^2)}, \quad b > 0.$$

In this case one can take $\Lambda \rightarrow \infty$ with g fixed and g_0 real. This phenomenon is known as asymptotic freedom since the bare coupling goes to 0 through real values as $\Lambda \rightarrow \infty$. These quantum field theories can be renormalized consistently in the weak coupling limit as long as one restricts one's attention to very large momenta. The other side of the coin is that they become strong coupling theories at low momenta, and so weak coupling approximations are useless for studying their particle spectrum.

Next we turn to the interpretation of the singularity in T or R as $k^2 \rightarrow -4m^2$. At this point the logarithm has a branch point, leading to a discontinuity depending on whether one goes past the branch point in the upper or lower half plane:

$$\begin{aligned} T_{\pm}(k^2) &= \frac{Q^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \frac{\Lambda^2}{|m^2 + x(1-x)k^2|} \\ &\mp i \frac{Q^2}{2\pi} \int_0^1 dx x(1-x) \theta(-x(1-x)k^2 - 4m^2) \\ &= \frac{Q^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \frac{\Lambda^2}{|m^2 + x(1-x)k^2|} \\ &\mp i \frac{Q^2}{2\pi} \int_0^1 dx x(1-x) \theta(-x(1-x)k^2 - 4m^2). \end{aligned}$$

The integral in the second term contributes only if $k^2 < -4m^2$ when the range of x contributing is

$$\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{m^2}{k^2}} < x < \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{m^2}{k^2}}.$$

The integral over that range is elementary and yields

$$\begin{aligned} T_{\pm}(k^2) &= \frac{Q^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \frac{\Lambda^2}{|m^2 + x(1-x)k^2|} \\ &\mp i \frac{Q^2}{2\pi} \frac{1}{3} \theta(-k^2 - 4m^2) \sqrt{\frac{1}{4} + \frac{m^2}{k^2}} \left(1 - \frac{2m^2}{k^2}\right). \end{aligned}$$

The $-i\epsilon$ prescription tells us to choose the lower half plane continuation, *i.e.* the lower (+) sign is to be taken.

$$\begin{aligned} T(k^2) &= \frac{Q^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \frac{\Lambda^2}{|m^2 + x(1-x)k^2|} \\ &\quad + i \frac{\alpha}{3} \theta(-k^2 - 4m^2) \sqrt{1 + \frac{4m^2}{k^2}} \left(1 - \frac{2m^2}{k^2}\right). \\ R(k^2) &= \frac{Q^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \frac{\Lambda^2}{|m^2 + x(1-x)k^2|} \\ &\quad + i \frac{\alpha_0}{3} \epsilon(k^0) \theta(-k^2 - 4m^2) \sqrt{1 + \frac{4m^2}{k^2}} \left(1 - \frac{2m^2}{k^2}\right), \end{aligned}$$

where we have used the definition of the bare fine structure constant $\alpha_0 = Q^2/4\pi$ and the relation between

R and T .

We can associate the appearance of an imaginary part of T with pair production which becomes energetically possible for energies larger than $2m$. To see the connection recall the variational equation satisfied by the vacuum persistence amplitude

$$\begin{aligned}\delta \ln \langle out|in \rangle &= i \int d^4 x \frac{\langle out|j^\mu(x)|in \rangle}{\langle out|in \rangle} \delta A_\mu(x) \\ &\approx - \int d^4 x d^4 y \langle 0, I | T[j_I^\mu(x) j_I^\nu(y)] | 0, I \rangle A_\nu(y) \delta A_\mu(x)\end{aligned}$$

where we used first order perturbation theory in the external field to approximate the current matrix element. The results of our calculation gave

$$i \langle 0, I | T[j_I^\mu(x) j_I^\nu(y)] | 0, I \rangle = \int \frac{d^4 k}{(2\pi)^4} e^{ik(x-y)} (k^\mu k^\nu - k^2 \eta^{\mu\nu}) T(k^2),$$

so we find, in weak field approximation,

$$\langle out|in \rangle \approx \exp \left\{ \frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} T(k^2) \tilde{A}_\mu(-k) (k^\mu k^\nu - k^2 \eta^{\mu\nu}) \tilde{A}_\nu(k) \right\}.$$

It is illuminating to express this in terms of the Fourier components of the field strengths

$$\begin{aligned}\tilde{F}_{\mu\nu}(k) &= i(k_\mu \tilde{A}_\nu(k) - k_\nu \tilde{A}_\mu(k)) \\ \tilde{F}_{\mu\nu}(-k) \tilde{F}^{\mu\nu}(k) &= -2(k^\mu k^\nu - k^2 \eta^{\mu\nu}) \tilde{A}_\mu(-k) \tilde{A}_\nu(k) \\ \langle out|in \rangle &\approx \exp \left\{ -\frac{i}{4} \int \frac{d^4 k}{(2\pi)^4} T(k^2) \tilde{F}_{\mu\nu}(-k) \tilde{F}^{\mu\nu}(k) \right\}.\end{aligned}$$

As long as T is real, this is a pure phase and $|\langle out|in \rangle| = 1$. But when T acquires an imaginary part as it does for $k^2 < -4m^2$ this is no longer true. Specifically,

$$|\langle out|in \rangle|^2 \approx \exp \left\{ \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \text{Im} T(k^2) \tilde{F}_{\mu\nu}(-k) \tilde{F}^{\mu\nu}(k) \right\}$$

which is the probability that there is *no* pair creation. The probability of pair creation is $1 - |\langle out|in \rangle|^2$ and is approximately given by

$$\begin{aligned}P_{pair} &\approx -\frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \text{Im} T(k^2) \tilde{F}_{\mu\nu}(-k) \tilde{F}^{\mu\nu}(k) \\ &\approx - \int \frac{d^4 k}{(2\pi)^4} \frac{\alpha}{6} \theta(-k^2 - 4m^2) \sqrt{1 + \frac{4m^2}{k^2}} \left(1 - \frac{2m^2}{k^2}\right) \tilde{F}_{\mu\nu}(-k) \tilde{F}^{\mu\nu}(k),\end{aligned}$$

where we have replaced the bare fine structure constant $\alpha_0 = \alpha(1 + R(0)) \rightarrow \alpha$ which is correct to the order we are calculating*.

* Strictly speaking, this calculation of $\langle out|in \rangle$ is for a fixed external field for which QA is fixed and finite. Renormalization applies only when the induced fields are included. In that case the \tilde{F} appearing in these formulas are the total fields, and for them $Q\tilde{F}$ is indeed held fixed as the cutoff is removed. That is, $\sqrt{1 + R(0)}\tilde{F}$ is the measured field strength.

Note that one could also find this result by calculating $|\langle out|b^{out}d^{out}|in\rangle|^2$ directly in perturbation theory and summing over all final states. It is of course important that this probability be positive (and $|\langle out|in\rangle| < 1$). Indeed, $\tilde{F}_{\mu\nu}(-k)\tilde{F}^{\mu\nu}(k)$ is negative for k timelike, because then there is a Lorentz frame where $\mathbf{k} = 0$ which implies $\mathbf{B} = 0$ so $\tilde{F}_{\mu\nu}(-k)\tilde{F}^{\mu\nu}(k) = -2|\tilde{\mathbf{E}}|^2$. Since it is an invariant, it must be negative in all frames.

When the fields have support only where T is real, the amplitude $\langle out|in\rangle$, even though it is a pure phase, gives information about the energy of the system in the presence of external fields. To get this connection consider a static field with adiabatic switching off at early and late times:

$$F_{\mu\nu}(x) = F_{\mu\nu}(\mathbf{x})\epsilon(t).$$

We take ϵ to be a symmetric function of t with central value 1 and gradual fall off to zero at times roughly $\pm T$. The exact shape is unimportant: we only need that $\tilde{\epsilon}(k^0)$ peaked at 0 with a width of order $1/T$. Clearly

$$\int dk^0 \tilde{\epsilon}(k^0)^2 = 2\pi \int dt \epsilon(t)\epsilon(-t) \approx 4\pi T.$$

Thus in the limit $T \rightarrow \infty$, $\tilde{\epsilon}^2$ can be approximated by $4\pi T\delta(k^0)$. Thus in this limit our approximate formula for $\langle out|in\rangle$ reads

$$\langle out|in\rangle \approx \exp \left\{ -2iT \int \frac{d^3k}{(2\pi)^3} \frac{1}{4} \tilde{F}_{\mu\nu}(-\mathbf{k})\tilde{F}^{\mu\nu}(\mathbf{k})T(\mathbf{k}^2) \right\}.$$

The coefficient of $-2iT$ is just the energy of the Dirac system in the presence of static external fields. Since T is positive for spacelike k , and

$$\frac{1}{4} \tilde{F}_{\mu\nu}(-\mathbf{k})\tilde{F}^{\mu\nu}(\mathbf{k}) = \frac{1}{2} [|\tilde{\mathbf{B}}|^2 - |\tilde{\mathbf{E}}|^2]$$

we see that the energy increases under the addition of a magnetic field and decreases with an electric field in accord with our conclusion that the vacuum is a diamagnetic dielectric medium.

Superconductivity and the Higgs Mechanism There is one physical phenomenon which can occur in quantum gauge field theories, although not for QED in four dimensions. It can happen in some theories that $R(k^2)$ possesses a pole at zero:

$$R(k^2) \underset{k^2 \rightarrow 0}{\sim} \frac{K}{k^2}.$$

Note that this infrared singularity does not occur in our previous calculation even for $m = 0$. But if it does occur, then

$$\tilde{A}(k)_{TOT} = \frac{1}{1 + R(k^2)} \tilde{A}_e \underset{k^2 \rightarrow 0}{\sim} \frac{k^2}{K} A_e \sim 0.$$

Thus A_{TOT} is screened at long wavelengths by the induced currents. In particular, for a static Coulomb potential $\tilde{A}^0 \sim \delta(k^0)/k^2$, the singularity at vanishing k disappears so in coordinate space the potential falls off faster than any power. Because of Lorentz covariance this screening is effective for *both* electric and magnetic fields. The screening of the magnetic fields means there is a Meissner effect, *i.e.* the vacuum in this situation is a relativistic superconductor. The vanishing of the total field at $k^2 = 0$ means that the vacuum cannot support massless photons. On the other hand there most likely is a negative value of k^2 call it $-M^2$ for which $1 + R(-M^2) = 0$. For such values of k , A_{TOT} can be nonzero even for vanishing external field. These waves correspond to particles of mass M . Thus this phenomenon, sometimes called the Higgs

mechanism, gives the photon a mass without violating gauge invariance. A nonrelativistic version of this effect was long known for superconductors. It was first discussed in the context of relativistic quantum field theory by Schwinger for QED in two space-time dimensions. Four dimensional versions were first discussed by Higgs, Englert and Brout, and Guralnik, Hagen and Kibble. In spite of the long list of discoverers, it seems that Higgs' name has stuck. The Higgs mechanism is at the heart of the electroweak unified gauge theory, because it is responsible for the masses of the W and Z bosons.