9. Second Quantized Dirac Equation

The formalism developed in the last chapter can now be fruitfully applied to the Dirac equation, cast as a Schrödinger equation,

$$i\frac{\partial\psi(\mathbf{x},t)}{\partial t} = \left(\frac{1}{i}\boldsymbol{\alpha}\cdot\nabla + \beta m\right)\psi(\mathbf{x},t).$$

Let us regard the component label a as an additional coordinate: $\psi_a(\mathbf{x}) = \psi(\mathbf{x},a)$. For definiteness let us choose our single particle basis to be momentum eigenstates, with an additional label λ for spin and $\pm$ to distinguish positive and negative energy states. Thus the role of $\psi_\alpha(\mathbf{x})$ of the previous chapter will be played by

$$\psi^{(\pm)}_{\lambda,\mathbf{p}}(\mathbf{x},a) = \frac{1}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p})}}u^a_{\lambda\pm}(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}.$$

The prefactors are conventional and with them in place the condition of orthonormality implies that

$$\sum_a u^{a*}_{\lambda'\pm'}u^a_{\lambda\pm} = 2\omega(\mathbf{p})\delta_{\lambda'\lambda}\delta_{\pm'\pm}.$$

We therefore write the explicit solution (7.6) for $u = u^+$ in the rescaled form

$$u_\lambda(\mathbf{p}) = \sqrt{\omega(\mathbf{p})+m}\begin{pmatrix}\phi_\lambda\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{m+\omega(\mathbf{p})}\phi_\lambda\end{pmatrix}$$

so that the normalization condition on the two spinor ϕ is

$$\phi^\dagger_{\lambda'}\phi_\lambda = \delta_{\lambda'\lambda}.$$

There are two widely used choices for ϕ_λ. One is to simply choose the two orthogonal spinors

$$\begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix}.$$

In the rest frame $\mathbf{p}=0$ these are just eigenstates of Σ_3 with eigenvalues $+1,-1$ respectively. The other choice is to pick them to be eigenstates of helicity $h = \mathbf{p}\cdot\boldsymbol{\sigma}/2|\mathbf{p}|$ denoted $\chi_\lambda(\mathbf{p})$:

$$h\chi_\lambda(\mathbf{p}) = \lambda\chi_\lambda(\mathbf{p}) \qquad\qquad \lambda = \pm\frac{1}{2}.$$

Explicit forms for χ_λ are developed in the exercises. Notice that for the helicity basis the expression for u simplifies to

$$u_\lambda(\mathbf{p}) = \sqrt{\omega(\mathbf{p})+m}\begin{pmatrix}\chi_\lambda\\ \frac{2\lambda|\mathbf{p}|}{m+\omega(\mathbf{p})}\chi_\lambda\end{pmatrix},$$

and furthermore u_λ is itself an eigenstate of helicity $\mathbf{p}\cdot\boldsymbol{\Sigma}/2|\mathbf{p}|$ with eigenvalue λ.

In all cases we maintain our choice (7.7)

$$u_-(\mathbf{p}) = i\gamma^2 u^*(-\mathbf{p})$$

for the negative energy basis functions. For the helicity basis choice for u_λ, this construction gives a negative energy spinor with the same helicity, as can easily be shown by applying h to both sides. For the rest frame Σ_3 basis this construction reverses the sign of Σ_3 in the rest frame.

The properties of $\psi^{(\pm)}_{\lambda,\mathbf{p}}(\mathbf{x},a)$ needed for the second quantization formalism are

$$\text{Orthonormality:} \qquad \int d^3x \sum_a \psi^{(\pm)*}_{\lambda,\mathbf{p}}(\mathbf{x},a)\psi^{(\pm)'}_{\lambda',\mathbf{p}'}(\mathbf{x},a) = \delta_{\lambda'\lambda}\delta_{(\pm)'\pm}\delta(\mathbf{p}'-\mathbf{p}) \tag{9.1}$$

and

$$\text{Completeness:} \qquad \int d^3p \sum_{\lambda\pm} \psi^{(\pm)}_{\lambda,\mathbf{p}}(\mathbf{x},a)\psi^{(\pm)*}_{\lambda,\mathbf{p}}(\mathbf{x}',a') = \delta_{aa'}\delta(\mathbf{x}'-\mathbf{x}). \tag{9.2}$$

To pass to the second quantized formalism, we simply introduce creation and annihilation operators $b^\dagger_{\lambda\pm}(\mathbf{p}), b_{\lambda\pm}(\mathbf{p})$ with anticommutation relations

$$\{b_{\lambda\pm}(\mathbf{p}), b^\dagger_{\lambda'\pm'}(\mathbf{p}')\} = \delta_{\lambda'\lambda}\delta_{(\pm)'\pm}\delta(\mathbf{p}'-\mathbf{p}),$$

and define the Dirac quantum field operator

$$\psi^a(\mathbf{x}) = \int d^3p \sum_{\lambda\pm} b_{\lambda\pm}(\mathbf{p})\psi^{(\pm)}_{\lambda,\mathbf{p}}(\mathbf{x},a).$$

By virtue of (9.2) the field operators satisfy

$$\{\psi^a(\mathbf{x}), \psi^{a'\dagger}(\mathbf{x}')\} = \delta_{aa'}\delta(\mathbf{x}-\mathbf{x}').$$

A principal virtue of second quantization is the efficiency with which we can construct the state describing the negative energy sea. We simply apply to the empty state all of the creation operators for negative energy states:

$$|sea\rangle \equiv |0\rangle = \mathcal{N}\prod_{\mathbf{p},\lambda} b^\dagger_{\lambda-}(\mathbf{p})\,|0'\rangle\,. \tag{9.3}$$

This looks terribly complicated, but we can uniquely characterize this state very simply: It is annihilated by all of the positive energy annihilation operators and by all the negative energy creation operators

$$b_{\lambda+}(\mathbf{p})\,|0\rangle = b^\dagger_{\lambda-}(\mathbf{p})\,|0\rangle = 0.$$

These conditions tell us everything we need to know about the sea. The annihilation operator for a negative energy electron creates a hole in the sea. Thus the construction of the sea is completely equivalent to interchanging the role of the creation and annihilation operators for the negative energy Dirac particles.

To see the consequences of this interchange of roles, let us consider a few of the observables of the theory. The Hamiltonian is just

$$\begin{aligned} H &= \int d^3x \sum_a \psi^{a\dagger}(\mathbf{x})(\frac{1}{i}\boldsymbol{\alpha}\cdot\nabla + \beta m)\psi^a(\mathbf{x}) \\ &= \int d^3p\omega(\mathbf{p}) \sum_\lambda (b^\dagger_{\lambda+}(\mathbf{p})b_{\lambda+}(\mathbf{p}) - b^\dagger_{\lambda-}(\mathbf{p})b_{\lambda-}(\mathbf{p})) \\ &= \int d^3p\omega(\mathbf{p}) \sum_\lambda (b^\dagger_{\lambda+}(\mathbf{p})b_{\lambda+}(\mathbf{p}) + b_{\lambda-}(\mathbf{p})b^\dagger_{\lambda-}(\mathbf{p})) - 2\int d^3p\omega(\mathbf{p})\delta(\mathbf{0}) \end{aligned}$$

where in the last form we have reordered the creation and annihilation operators of the negative energy contributions, the nonzero anticommutator producing the negative infinite constant term. Notice that thanks to the Fermi statistics both contributions to the energy are positive. This constant is just the energy of the sea. the factor of $\delta(\mathbf{0})$ can be identified with $Volume/(2\pi)^3$ so the sea has an infinite negative energy density. As we have already stressed we can and will choose to measure all energies relative to that of the sea which amounts to dropping this constant*, so henceforth we shall take the free Dirac second quantized Hamiltonian to be

$$H_{Dirac} = \int d^3p\ \omega(\mathbf{p}) \sum_\lambda (b^\dagger_{\lambda+}(\mathbf{p})b_{\lambda+}(\mathbf{p}) + b_{\lambda-}(\mathbf{p})b^\dagger_{\lambda-}(\mathbf{p})).$$

When we pass to the Heisenberg picture we find the field equation for ψ to be nothing other than the Dirac equation

$$(\frac{1}{i}\gamma\cdot\partial + m)\psi = 0.$$

Since we have selected our single particle basis to be eigenstates of $\frac{1}{i}\boldsymbol{\alpha}\cdot\nabla + \beta m$ the time dependence of ψ in Heisenberg picture is simply

$$\psi^a(\mathbf{x},t) = \int d^3p \sum_{\lambda\pm} b_{\lambda\pm}(\mathbf{p})\psi^{(\pm)}_{\lambda,\mathbf{p}}(\mathbf{x},a)e^{\mp i\omega(\mathbf{p})t} \tag{9.4}$$

The annihilation operators for the positive and negative energy Dirac particles are thus identified with the positive and negative frequency components of the Dirac field in Heisenberg picture. This is a useful observation because when we introduce time dependent external fields which are switched off at early and late times, it will allow us to easily relate the operators that characterize the sea at late times to the ones that characterize the sea at early times.

Returning to our survey of observables, the momentum operator is just

$$\begin{aligned}\mathbf{P} &= \int d^3x \sum_a \psi^{a\dagger}(\mathbf{x})\frac{1}{i}\nabla\psi^a(\mathbf{x})\\ &= \int d^3p\mathbf{p} \sum_\lambda (b^\dagger_{\lambda+}(\mathbf{p})b_{\lambda+}(\mathbf{p}) + b^\dagger_{\lambda-}(\mathbf{p})b_{\lambda-}(\mathbf{p}))\\ &= \int d^3p\mathbf{p} \sum_\lambda (b^\dagger_{\lambda+}(\mathbf{p})b_{\lambda+}(\mathbf{p}) + b_{\lambda-}(-\mathbf{p})b^\dagger_{\lambda-}(-\mathbf{p}))\end{aligned}$$

The term $2\int d^3p\mathbf{p}\delta(\mathbf{0})$ arising from reordering the negative energy operators automatically vanishes and need not be dropped. We see from the explicit form of the momentum operator that $b_{\lambda-}(-\mathbf{p})$ creates from the sea a particle of momentum $+\mathbf{p}$.

The charge operator is just $Q = qN$ where q is the unit of charge carried by the Dirac particle and N is the number operator

$$\begin{aligned}Q =& q\int d^3x \sum_a \psi^{a\dagger}(\mathbf{x})\psi^a(\mathbf{x})\\ =& q\int d^3p \sum_\lambda (b^\dagger_{\lambda+}(\mathbf{p})b_{\lambda+}(\mathbf{p}) - b_{\lambda-}(-\mathbf{p})b^\dagger_{\lambda-}(-\mathbf{p})) + 2q\int d^3p\delta(\mathbf{0})\end{aligned}$$

from which we see that $b_{\lambda-}(-\mathbf{p})$ creates a state of charge $-q$. We shall also in future drop the constant term

* Alternatively we could introduce a bare cosmological constant to cancel it.

in Q so the charge of the sea is then zero[†] There is a convenient way to make this subtraction. Instead of taking the charge density to be $\rho(\mathbf{x}) = q\psi^\dagger\psi$, take it to be the symmetrized form

$$\rho(\mathbf{x}) = \frac{q}{2}\sum_a [\psi_a^\dagger(\mathbf{x})\psi_a(\mathbf{x}) - \psi_a(\mathbf{x})\psi_a^\dagger(\mathbf{x})] \tag{9.5}$$

Then when the operators in $Q = \int d^3x\rho$ are suitably reordered the piece coming from the positive energy term exactly cancels that from the negative energy term. At this point we can also identify the current operator from local current conservation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0. \tag{9.6}$$

Inserting (9.5) into (9.6) and using the Dirac equation, we identify

$$\mathbf{j} = \frac{q}{2}\sum_a [\psi_a^\dagger(\mathbf{x})(\boldsymbol{\alpha}\psi)_a(\mathbf{x}) - (\boldsymbol{\alpha}\psi)_a(\mathbf{x})\psi_a^\dagger(\mathbf{x})].$$

We can assemble $(\rho, \mathbf{j})$ in a four vector j^μ:

$$\begin{aligned} j^\mu(\mathbf{x},t) =& \frac{q}{2}\sum_a [\psi_a^\dagger(\mathbf{x},t)(\beta\gamma^\mu\psi)_a(\mathbf{x},t) - (\beta\gamma^\mu\psi)_a(\mathbf{x},t)\psi_a^\dagger(\mathbf{x},t)] \\ =& \frac{q}{2}\sum_a [\psi_a^\dagger(\mathbf{x},t), (\beta\gamma^\mu\psi)_a(\mathbf{x},t)] \\ =& \frac{q}{2}\sum_a [\bar\psi_a(\mathbf{x},t), (\gamma^\mu\psi)_a(\mathbf{x},t)] \end{aligned}$$

We have made use of the Dirac adjoint

$$\bar\psi_a \equiv \sum_b \psi_b^\dagger\beta_{ba}$$

in the last form, which we will also sometimes shorten even more by suppressing the spinor indices, $\frac{q}{2}[\bar\psi, \gamma^\mu\psi]$. Current conservation $\partial_\mu j^\mu = 0$ is an immediate consequence of the Dirac equation and its Dirac adjoint

$$\bar\psi(-i\gamma\cdot\overleftarrow{\partial} - m) = 0.$$

The final observable we mention is the angular momentum

$$\mathbf{J} = \int d^3x \sum_{a,b}\psi^{a\dagger}(\mathbf{x})\left(\frac{1}{i}(\mathbf{x}\times\nabla) + \frac{1}{2}\boldsymbol{\Sigma}\right)_{ab}\psi^b(\mathbf{x}).$$

The sea is of course rotationally invariant

$$\mathbf{J}\,|0\rangle = 0,$$

as will be shown in an exercise. Of particular interest is the action of the helicity on the single particle states.

† If there is more than one species of fermion in the universe, the coefficient of this term is $\sum_f q_f$. One way of getting rid of the sea charge is to insist that this sum of charges vanishes. As it happens the standard model of strong weak and electromagnetic interactions has this property, which is required for cancellation of the axial anomaly. Thanks to Mr. Yan-Bo Xie for drawing my attention to this circumstance. In a similar vein supersymmetry is often proposed so that the zero point energy of bosons exactly cancels the sea energy of fermions.

On the particle states we find

$$\mathbf{p}\cdot\mathbf{J}b^\dagger_{\lambda+}(\mathbf{p})\left|0\right\rangle = \frac{1}{2\omega}\sum_a u^{a*}_{\lambda'+}(\mathbf{p})\frac{1}{2}\mathbf{p}\cdot\mathbf{\Sigma}_{ab}u^b_{\lambda+}(\mathbf{p})b^\dagger_{\lambda'+}(\mathbf{p})\left|0\right\rangle = \lambda|\mathbf{p}|b^\dagger_{\lambda+}(\mathbf{p})\left|0\right\rangle,$$

confirming that this state carries helicity λ. The helicity of the one hole state $b_{\lambda-}(-\mathbf{p})\left|0\right\rangle$ which possesses momentum $+\mathbf{p}$ is also λ but the reason is slightly subtle. First of all

$$\mathbf{p}\cdot\mathbf{J}b_{\lambda-}(-\mathbf{p})\left|0\right\rangle = -\frac{1}{2\omega}\sum_a u^{a*}_{\lambda-}(-\mathbf{p})\frac{1}{2}\mathbf{p}\cdot\mathbf{\Sigma}_{ab}u^b_{\lambda'-}(-\mathbf{p})b_{\lambda'-}(-\mathbf{p})\left|0\right\rangle$$

where the minus sign arises because $b^\dagger_{\lambda-}$ occurs in $\psi^\dagger$ and must anticommute with ψ before it can contract against $b_{\lambda-}$. But then

$$\frac{1}{2}(-\mathbf{p})\cdot\mathbf{\Sigma}u_{\lambda-}(-\mathbf{p}) = \lambda|\mathbf{p}|u_{\lambda-}(-\mathbf{p}).$$

Thus

$$\frac{\mathbf{p}\cdot\mathbf{J}}{|\mathbf{p}|}b_{\lambda-}(-\mathbf{p})\left|0\right\rangle = \lambda b_{\lambda-}(-\mathbf{p})\left|0\right\rangle$$

as we claimed.

This survey of single particle observables has established:

6. The state $b^\dagger_{\lambda+}(\mathbf{p})\left|0\right\rangle$ is a one particle state of momentum $\mathbf{p}$, energy $\omega=\sqrt{\mathbf{p}^2+m^2}$, charge q, and helicity λ.
7. The state $b_{\lambda-}(-\mathbf{p})\left|0\right\rangle$ is a one particle state of momentum $\mathbf{p}$, energy $\omega=\sqrt{\mathbf{p}^2+m^2}$, charge $-q$, and helicity λ.

In particular if the first state is an electron of charge $-e$, then the second is a positron of charge $+e$. To emphasize these facts it is traditional to rename the creation and annihilation operators for negative energy particles. So define

$$b_{\lambda-}(-\mathbf{p})\equiv d^\dagger_\lambda(\mathbf{p}) \qquad\qquad b^\dagger_{\lambda+}(\mathbf{p})\equiv b^\dagger_\lambda(\mathbf{p}).$$

So $b^\dagger_\lambda(\mathbf{p})$ creates a particle and $d^\dagger_\lambda(\mathbf{p})$ creates an antiparticle. Similarly it is useful to define the Dirac spinor

$$v_\lambda(\mathbf{p})\equiv u_{\lambda-}(-\mathbf{p}) = i\gamma^2u^*_\lambda(\mathbf{p}).$$

Note that u and v satisfy

$$\begin{aligned}(\gamma\cdot p+m)u_\lambda(\mathbf{p}) &= 0 \qquad\qquad (h-\lambda)u_\lambda(\mathbf{p})=0\\ (\gamma\cdot p-m)v_\lambda(\mathbf{p}) &= 0 \qquad\qquad (h+\lambda)v_\lambda(\mathbf{p})=0\end{aligned}$$

with opposite signs in front of the mass and opposite helicities.

With these definitions the free Dirac field operator in Heisenberg picture has the representation

$$\psi(x) = \int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega}}\sum_\lambda\left(b_\lambda(\mathbf{p})u_\lambda(\mathbf{p})e^{ix\cdot p}+d^\dagger_\lambda(\mathbf{p})v_\lambda(\mathbf{p})e^{-ip\cdot x}\right),$$

where $p\cdot x=\mathbf{p}\cdot\mathbf{x}-\omega(\mathbf{p})t$ is the Minkowski scalar product. A point to bear in mind with this new interpretation is that one body operators will generally contain terms like $b^\dagger d^\dagger$ which create a particle antiparticle pair and terms like bd which destroy such a pair. Thus when we couple couple currents to the electromagnetic field we will have charge conservation, but *not* particle number conservation.

10. The Discrete Symmetries of the Dirac Equation

10.1. Parity

The parity transformation $\mathbf{x} \to -\mathbf{x}$ can be extended to a symmetry of the Hamiltonian. Consider the following transformation on the field:

$$\psi(\mathbf{x},t) \to P^{-1}\psi(\mathbf{x},t)P = e^{i\phi}\beta\psi(-\mathbf{x},t), \tag{10.1}$$

where we have allowed a multiplicative phase. Then the Hamiltonian transforms to

$$\begin{aligned} P^{-1}HP &= \int d^3x\psi^\dagger(-\mathbf{x})\beta(\frac{1}{i}\boldsymbol{\alpha}\cdot\nabla + \beta m)\beta\psi(-\mathbf{x}) \\ &= \int d^3x\psi^\dagger(-\mathbf{x})(-\frac{1}{i}\boldsymbol{\alpha}\cdot\nabla + \beta m)\psi(-\mathbf{x}) = H \end{aligned} \tag{10.2}$$

after changing integration variables, so it is parity invariant. From the parity transformation (10.1) we can infer how Parity acts on the particle states:

$$e^{i\phi}\beta\psi(-\mathbf{x},t) = e^{i\phi}\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega}}\sum_\lambda\left(b_\lambda(-\mathbf{p})\beta u_\lambda(-\mathbf{p})e^{ix\cdot p} + d^\dagger_\lambda(-\mathbf{p})\beta v_\lambda(-\mathbf{p})e^{-ip\cdot x}\right),$$

where we have reversed the sign of $\mathbf{p}$ by a change of variables. Next we note that depending on the spin basis we choose,

$$\text{Helicity Basis:}\qquad \begin{aligned} \beta u_\lambda(\mathbf{p}) &= -\,ie^{i\lambda(\pi+2\phi_\mathbf{p})}u_{-\lambda}(-\mathbf{p}) \\ \beta v_\lambda(\mathbf{p}) &= -\,ie^{-i\lambda(\pi+2\phi_\mathbf{p})}v_{-\lambda}(-\mathbf{p}) \end{aligned}$$

where we have used the formula

$$\chi_\lambda(\mathbf{p}) = -ie^{i\lambda(\pi+2\phi_\mathbf{p})}\chi_{-\lambda}(-\mathbf{p})$$

obtained in the exercises, or

$$\text{Rest Frame }\Sigma_3:\qquad \begin{aligned} \beta u_\mu(\mathbf{p}) &= u_\mu(-\mathbf{p}) \\ \beta v_\mu(\mathbf{p}) &= -\,v_\mu(-\mathbf{p}). \end{aligned}$$

Using these spinor properties, we learn that

$$\text{Helicity Basis:}\qquad \begin{aligned} P^{-1}b_\lambda(\mathbf{p})P &= e^{i\phi}ie^{-i\lambda(\pi+2\phi_\mathbf{p})}b_{-\lambda}(-\mathbf{p}) \\ P^{-1}d^\dagger_\lambda(\mathbf{p})P &= e^{i\phi}ie^{i\lambda(\pi+2\phi_\mathbf{p})}d^\dagger_{-\lambda}(-\mathbf{p}) \\ P^{-1}b^\dagger_\lambda(\mathbf{p})P &= e^{-i\phi}(-i)e^{i\lambda(\pi+2\phi_\mathbf{p})}b^\dagger_{-\lambda}(-\mathbf{p}) \end{aligned}$$

where the last equation is just the hermitian conjugate of the first. And

$$\text{Rest Frame }\Sigma_3:\qquad \begin{aligned} P^{-1}d^\dagger_\mu(\mathbf{p})P &= -\,e^{i\phi}d^\dagger_\mu(-\mathbf{p}) \\ P^{-1}b^\dagger_\mu(\mathbf{p})P &= e^{-i\phi}b^\dagger_\mu(-\mathbf{p}). \end{aligned}$$

It should be noted that the arbitrary phase we allowed in the definition of parity cancels out for neutral states, *i.e.* those with an equal number of $b^\dagger$'s and $d^\dagger$'s acting on the sea, which we can take to be parity invariant. This means that whereas the intrinsic parity of a single particle is conventional, that of a particle antiparticle pair is not. For example the above formulae imply that the parity of the ground state s wave of positronium is odd, *i.e.* the ground states are 0^- and 1^-.

It is apparent that a Dirac particle and its antiparticle are closely related to each other. They have identical mass and spin, but exactly opposite charges. In fact this relationship is a reflection of a symmetry in the dynamics under interchange of a particle with its antiparticle. To explore this symmetry, first define a unitary transformation C, $CC^\dagger = I$ by the rules

$$C^{-1}b^\dagger C = d^\dagger \qquad C^{-1}d^\dagger C = b^\dagger \qquad C\left|0\right\rangle = \left|0\right\rangle. \tag{10.3}$$

(The last of these equations implies an extremely complicated transformation of the empty state, but we shall never see those complications.) From the definition of C we can work out how the field transforms

$$C^{-1}\psi(x)C = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega}} \sum_\lambda \left(d_\lambda(\mathbf{p})u_\lambda(\mathbf{p})e^{ix\cdot p} + b_\lambda^\dagger(\mathbf{p})v_\lambda(\mathbf{p})e^{-ip\cdot x}\right)$$

which we can relate to $\psi^\dagger$:

$$\psi_a^\dagger(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega}} \sum_\lambda \left(b_\lambda^\dagger(\mathbf{p})u_{\lambda a}^*(\mathbf{p})e^{-ix\cdot p} + d_\lambda(\mathbf{p})v_{\lambda a}^*(\mathbf{p})e^{ip\cdot x}\right).$$

But $v^* = i\gamma^2 u$ and $u^* = i\gamma^2 v$ so we can infer

$$C^{-1}\psi_a(x)C = i(\gamma^2)_{ab}\psi_b^\dagger(x). \tag{10.4}$$

When we suppress spinor indices, it is usually convenient to think of $\psi^\dagger$ as a row vector and ψ as a column vector. To write (10.4) with suppressed indices we want the r.h.s. to be a column vector which we could indicate by $(\psi^\dagger)^T$ meaning the transpose on spinor indices. In that case (10.4) can be written

$$C^{-1}\psi(x)C = i\gamma^2(\psi^\dagger)^T(x). \tag{10.5}$$

It should also be noted that the occurrence of γ^2 in the charge conjugation transformation law is specific to the standard representation; with other representations a different matrix would appear.

The invariance of H is obvious from its expression in terms of creation and annihilation operators, but it is also instructive to see it using the local definition

$$\begin{aligned}C^{-1}HC =& \int d^3x\psi^T i\gamma^2(\frac{1}{i}\boldsymbol{\alpha}\cdot\nabla + \beta m)i\gamma^2(\psi^\dagger)^T \\ =& -\int d^3x\psi^T(-\frac{1}{i}\boldsymbol{\alpha}^T\cdot\nabla + \beta^T m)(\psi^\dagger)^T \\ =& H\end{aligned}$$

where the last step involves an integration by parts, a transposition of Dirac indices, and a reordering of the order of ψ and $\psi^\dagger$ giving a minus sign which cancels the overall minus sign in the second line.

The transformation of the charge and current densities under C should simply change their signs. This is not hard to see:

$$\begin{aligned}C^{-1}j^\mu C =& \frac{q}{2}(\psi^T i\gamma^2\beta\gamma^\mu i\gamma^2(\psi^\dagger)^T - \psi^\dagger i\gamma^2\gamma^{\mu T}\beta i\gamma^2\psi \\ =& -j^\mu\end{aligned}$$

where use is made of

$$i\gamma^2\gamma^\mu i\gamma^2 = -\gamma^{\mu *} \qquad \beta\gamma^\mu\beta = \gamma^{\mu\dagger}.$$

Note that because we have used the symmetrized definition of the current, there is no reordering of operators necessary in arriving at this result.

In our discussion so far it has seemed inevitable that the Dirac particle carries charge. More precisely, it carries a conserved fermion number N which could be identified with charge. If there were several species f of Dirac particle the fermion number N_f of each species might be separately conserved. Including terms of the form

$$\bar{\psi}_f \Gamma \psi_{f'}$$

with $f' \neq f$ in the Hamiltonian could violate the individual N_f but $\sum_f N_f$ would still be conserved. Majorana pointed out that even with only one species of fermion it is possible to make it totally neutral, *i.e.* carry no conserved quantum number at all.

Starting with the Dirac theory we can see that this is possible, because we can consider redefining creation and annihilation operators to be eigenoperators under charge conjugation:

$$b_{\lambda\pm}(\mathbf{p}) = \frac{1}{\sqrt{2}}(b_\lambda(\mathbf{p}) \pm d_\lambda(\mathbf{p})) \qquad \text{with } C^{-1} b_{\lambda\pm}(\mathbf{p}) C = \mp b_{\lambda\pm}(\mathbf{p}).$$

Then the Hamiltonian is the sum of two commuting pieces

$$H = H_+ + H_- \qquad \text{with } H_\pm = \int d^3p\, \omega(\mathbf{p}) \sum_\lambda b^\dagger_{\lambda\pm}(\mathbf{p}) b_{\lambda\pm}(\mathbf{p}).$$

Clearly, it is perfectly consistent to consider the quantum system defined by H_+ (or H_-) alone. The number operator of the Dirac theory

$$N = \int d^3p \sum_\lambda (b^\dagger_{\lambda-}(\mathbf{p}) b_{\lambda+}(\mathbf{p}) + b^\dagger_{\lambda+}(\mathbf{p}) b_{\lambda-}(\mathbf{p})),$$

clearly has no meaning in the truncated theory, but that is to be expected.

One might worry that truncating the theory in this way might spoil locality, but this is not the case. We can just as easily redefine the local fields to be eigenoperators of charge conjugation:

$$\psi_\pm(\mathbf{x}) = \frac{1}{\sqrt{2}}(\psi(\mathbf{x}) \pm i\gamma^2 (\psi^\dagger)^T(\mathbf{x})) \qquad \text{with } C^{-1}\psi_\pm C = \pm\psi_\pm, \tag{10.6}$$

Which satisfy anticommutation relations

$$\{\psi^a_\pm(\mathbf{x}), \psi^b_\pm(\mathbf{x}')\} = \pm(i\gamma^2)_{ab}\delta(\mathbf{x} - \mathbf{x}').$$

Clearly

$$\psi_\pm(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega}} \sum_\lambda \left(b_{\lambda\pm}(\mathbf{p}) u_\lambda(\mathbf{p}) e^{ix\cdot p} \pm b^\dagger_{\lambda\pm}(\mathbf{p}) v_\lambda(\mathbf{p}) e^{-ip\cdot x} \right),$$

and in terms of these fields

$$\begin{aligned} H_\pm =& \frac{1}{2}\int d^3x \psi^\dagger_\pm (\frac{1}{i}\boldsymbol{\alpha}\cdot\nabla + \beta m)\psi_\pm \\ =& \frac{1}{2}\int d^3x (\pm\psi^T_\pm) i\gamma^2 (\frac{1}{i}\boldsymbol{\alpha}\cdot\nabla + \beta m)\psi_\pm. \end{aligned}$$

In the second line we have used $\psi^\dagger_\pm = \pm\psi^T_\pm i\gamma^2$, a consequence of (10.6).

The appearance of $i\gamma^2$ in the above discussion is due to our choice of the standard representation for the gamma matrices. The Majorana representation is characterized by the condition that the gamma matrices be pure imaginary $\gamma^{\mu *} = -\gamma^\mu$. In that case the charge conjugation transformation does not involve a matrix at all and all of the $i\gamma^2$'s disappear.

In the case of massless fermions $m = 0$, it is possible to describe relativistic spin $1/2$ particles with only one helicity. In the Dirac theory the easiest way to see this is to consider the matrix

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3,$$

which anticommutes with the γ^μ. It therefore commutes with the Lorentz matrices

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu].$$

γ_5 commutes with the $\boldsymbol{\alpha} \cdot \nabla$ term of the Dirac Hamiltonian but not with the $m\beta$ term. So when $m = 0$, and only then, the energy eigenstates can be simultaneously eigenstates of γ_5. Since $\gamma_5^2 = 1$ the eigenvalues of γ_5, called chirality, are ± 1 and

$$\frac{I \pm \gamma_5}{2}$$

are projectors onto orthogonal two dimensional subspaces with chirality ± 1 respectively. Defining

$$R = \frac{I + \gamma_5}{2}\psi \qquad\qquad L = \frac{I - \gamma_5}{2}\psi,$$

R for "right-handed" and L for "left-handed," the Dirac hamiltonian for $m = 0$ decomposes into two commuting terms

$$H = \int d^3x (R^\dagger \frac{1}{i}\boldsymbol{\alpha} \cdot \nabla R + L^\dagger \frac{1}{i}\boldsymbol{\alpha} \cdot \nabla L)$$

either of which could define a consistent dynamics of the corresponding subsystem. These subsystems are called Weyl fermions. The corresponding momentum space spinors of definite chirality are

$$\frac{I \pm \gamma_5}{2} u_\lambda(\mathbf{p}) = \frac{1}{2}\sqrt{|\mathbf{p}|}(1 \pm 2\lambda)\begin{pmatrix} \chi_\lambda \\ \pm\chi_\lambda \end{pmatrix},$$

from which it is clear that helicity is identical to $Chirality/2$. In other words right-handed Weyl fermions have helicity $+1/2$ and left-handed ones have helicity $-1/2$. Since γ_5 is real and γ^2 anticommutes with γ_5, the antiparticle spinors

$$\frac{I \pm \gamma_5}{2} v_\lambda(\mathbf{p}) = i\gamma^2 (\frac{I \pm \gamma_5}{2} u_\lambda(\mathbf{p}))^*$$

have the opposite correlation between chirality and helicity. So if the Weyl particle is right handed the particle has helicity $+1/2$ and the antiparticle has helicity $-1/2$. Since charge conjugation interchanges the role of particle and antiparticle one can even choose by convention all Weyl particles to be left(right)-handed. One can do this, for example, by writing $R^\dagger = L'^T i\gamma^2$. Of course, such an L' has charge opposite to R.

The Weyl particle with helicity $\pm 1/2$ comes along with its antiparticle with helicity $\mp 1/2$. Thus the Weyl system has the same helicity content as the massless Majorana system. In fact one can describe the Majorana system using Weyl fields. First, separate the Majorana field $\psi^\dagger = \psi^T i\gamma^2$ into two fields of definite chirality

$$\psi_{R,L} = \frac{I \pm \gamma_5}{2}\psi.$$

Then notice that

$$\begin{aligned}\psi_R^\dagger =& \psi^\dagger \frac{I+\gamma_5}{2} \\ =& \psi^T i\gamma^2 \frac{I+\gamma_5}{2} \\ =& \psi_L^T i\gamma^2\end{aligned}$$

so that the right-handed component of the Majorana field can be eliminated in favor of the hermitian conjugate of the left-handed component. When this is done the Majorana hamiltonian simplifies to

$$H_{Maj} = \int d^3x \psi_L^\dagger \frac{1}{i}\boldsymbol{\alpha} \cdot \nabla \psi_L + \frac{m}{2}\int d^3x [\underbrace{\psi_L^T i\gamma^2 \beta \psi_L}_{\Delta F=-2} + \underbrace{(\psi_L^T i\gamma^2\beta\psi_L)^\dagger}_{\Delta F=+2}]$$

which reduces to the Weyl hamiltonian for $m = 0$. Notice that the massless limit conserves fermion number but the mass term violates fermion number conservation by ± 2 units. This is the so-called Majorana mass term for a Weyl fermion. To construct the Dirac mass term which conserves fermion number, one must add a right-handed Weyl fermion to the theory.

The Weyl theory violates parity invariance, essentially because β fails to commute with γ_5:

$$P^{-1}L(\mathbf{x})P = \frac{I-\gamma_5}{2}\beta\psi(-\mathbf{x}) = \beta R(-\mathbf{x}).$$

How then have we managed to show that it is equivalent to the parity conserving Majorana theory? The answer is that although parity is violated in the Weyl theory, CP the product of parity times charge conjugation remains a symmetry. The Majorana field is inert under charge conjugation, and has only parity as a nontrivial symmetry. It is the CP symmetry of the Weyl theory that corresponds to the parity of the Majorana theory.

Notice that it is impossible to have a fermion that is simultaneously Majorana and Weyl: even at zero mass one always must have both helicities. This is true in four space-time dimensions but in other dimensions it need not be so. For example in 10 dimensions one can define Majorana-Weyl fermions. A Dirac fermion in $D = 2k$ dimensions has 2^k degrees of freedom, 2^{k-1} states for the particle and the same number for the antiparticle. In some dimensions (including 4 and 10) one can have Majorana fermions with only 2^{k-1} degrees of freedom. In the massless case one can define Weyl fermions in all even dimensions giving 2^{k-1} degrees of freedom. In $2 + 8n$ dimensions one can have Majorana-Weyl fermions with only 2^{k-2} degrees of freedom. For example in 10 dimensions a Dirac fermion has 32 states, a Majorana or Weyl fermion has 16 states, and a Majorana-Weyl fermion has only 8 states. This possibility is crucial for the consistency of superstring theory.

10.5. Time Reversal

The last discrete symmetry we discuss is time reversal T. It is well-known that T must be an antiunitary transformation, meaning that it is antilinear, and furthermore

$$\langle T\Phi|T\Psi\rangle = \langle\Psi|\Phi\rangle .$$

With this in mind, we search for a transformation of the form

$$T^{-1}\psi(\mathbf{x},t)T = \mathcal{T}\psi(\mathbf{x},-t),$$

with $\mathcal{T}$ an appropriate matrix. From antiunitarity we have

$$\left\langle\Phi|T^{-1}\psi T\Psi\right\rangle = \langle\psi T\Psi|T\Phi\rangle = \left\langle T\Psi|\psi^\dagger T\Phi\right\rangle = \left\langle T^{-1}\psi^\dagger T\Phi|\Psi\right\rangle ,$$

from which it follows that

$$T^{-1}\psi^\dagger(t)T = (T^{-1}\psi T)^\dagger = \psi^\dagger(-t)\mathcal{T}^\dagger.$$

Thus

$$T^{-1}HT = \int d^3x\psi^\dagger(\mathbf{x},-t)\mathcal{T}^\dagger\left(-\frac{1}{i}\boldsymbol{\alpha}^*\cdot\nabla+\beta m\right)\mathcal{T}\psi(\mathbf{x},-t).$$

If we choose $\mathcal{T}$ to be unitary, then invariance of H will be achieved (given conservation of energy $dH/dt=0$, which follows from the field equation) if and only if $\mathcal{T}$ commutes with β and α^2 and anticommutes with α^1 and α^3. Clearly the most general solution of these conditions is

$$\mathcal{T} = e^{i\tau}\gamma^1\gamma^3 = ie^{i\tau}\Sigma_2,$$

so that the transformation law becomes

$$T^{-1}\psi(\mathbf{x},t)T = ie^{i\tau}\Sigma_2\psi(\mathbf{x},-t). \tag{10.7}$$

This transformation law on ψ implies that for b,d. It is easiest to do this for the rest frame Σ_3 basis, because then the two-spinors ϕ_μ are real. Thus

$$\begin{aligned} u_\mu^*(-\mathbf{p}) =& \sqrt{\omega(\mathbf{p})+m}\begin{pmatrix}\phi_\mu\\ \frac{-\boldsymbol{\sigma}^*\cdot\mathbf{p}}{m+\omega(\mathbf{p})}\phi_\mu\end{pmatrix}\\ =&\Sigma_2\sqrt{\omega(\mathbf{p})+m}\begin{pmatrix}\sigma_2\phi_\mu\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{m+\omega(\mathbf{p})}\sigma_2\phi_\mu\end{pmatrix}\\ =& i^{2\mu}\Sigma_2 u_{-\mu}(\mathbf{p})\end{aligned} \tag{10.8}$$

Using $v = i\gamma^2u^*$, it is just a few steps to show that

$$v_\mu^*(-\mathbf{p}) = i^{2\mu}\Sigma_2 v_{-\mu}(\mathbf{p}). \tag{10.9}$$

Because T is antilinear, the l.h.s. of (10.7) involves $u^*e^{-ix\cdot p}T^{-1}bT$ and $v^*e^{+ix\cdot p}T^{-1}d^\dagger T$, so (10.8) and (10.9) allow us to infer from (10.7) that

$$\begin{aligned} T^{-1}b_\mu(\mathbf{p})T =& ie^{i\tau}i^{-2\mu}b_{-\mu}(-\mathbf{p}) \qquad & T^{-1}b_\mu^\dagger(\mathbf{p})T = -ie^{-i\tau}i^{+2\mu}b_{-\mu}^\dagger(-\mathbf{p})\\ T^{-1}d_\mu^\dagger(\mathbf{p})T =& ie^{i\tau}i^{-2\mu}d_{-\mu}^\dagger(-\mathbf{p}) \qquad & T^{-1}d_\mu(\mathbf{p})T = -ie^{-i\tau}i^{+2\mu}d_{-\mu}(-\mathbf{p})\end{aligned}$$

The reversal of signs of momentum and spin label is intuitively correct since time reversing a motion reverses both momentum and angular momentum. The μ dependence of the phase is perhaps less intuitive, but follows straightforwardly by using angular momentum raising and lowering operators together with the action of time reversal on angular momentum. If we had used the helicity basis, the helicity label would not be reversed by time reversal (remember it is $\mathbf{J}\cdot\mathbf{p}/|\mathbf{p}|$); unfortunately the phases that are induced are angle dependent and not very illuminating.

10.6. Violation of the Discrete Symmetries and the CPT Theorem

Having shown that C, P, and T are symmetries of the Dirac equation it is instructive to contemplate how things must be changed to violate these symmetries. For example, we have seen that both parity and charge conjugation are violated with Weyl fermions, but in such a way that CP remains a symmetry. More generally, one can consider adding noninvariant terms to the Hamiltonian. In exercises, it is shown how the bilinears $\bar\psi_A\Gamma\psi_B$ transform under these symmetries for $\Gamma=(I,i\gamma_5,\gamma^\mu,\gamma_5\gamma^\mu,\sigma^{\mu\nu})$. Under parity they transform with a factor of $(+,-,+,-,+)(-)^S$ times the bilinear evaluated with $\mathbf{x}\to-\mathbf{x}$, and where S is the number of *spatial* indices in the tensor component. So, examples of parity odd Lorentz invariants would be

$$\bar\psi i\gamma_5\psi \qquad\qquad \bar\psi\gamma^\mu\psi\bar\psi\gamma_5\gamma_\mu\psi. \tag{10.10}$$

Adding such terms to the energy density would appear to violate parity. One must be careful that the violation is not an illusion. For example the term $\psi^\dagger\boldsymbol{\alpha}\cdot\nabla\psi$ is invariant under the chiral symmetry

$$\psi\to e^{i\alpha\gamma_5}\psi$$

under which

$$\bar\psi\psi\to\cos 2\alpha\bar\psi\psi+\bar\psi i\gamma_5\psi\sin 2\alpha$$

so the added term $\bar\psi i\gamma_5\psi$ can be rotated into $\bar\psi\psi$ and parity violation disappears.

With regard to charge conjugation, $\bar\psi_A\Gamma\psi_B$ transforms to $(+,+,-,+,-)$ times $\bar\psi_B\Gamma\psi_A$. For example the first of (10.10) is invariant under charge conjugation (if $A=B$) but the second is odd. Under CP the result is $(+,-,-,-,-)(-)^S$. Thus it is the first of (10.10) that would violate CP. Since such a term can be rotated away by a chiral transformation we see that CP is a bit tricky to violate. For example in the Standard Model one needs at least three generations of quarks and leptons to frustrate the ability to transform away apparently CP violating couplings! Fortunately there is solid evidence for this number of generations.[⋆]

Finally we come to time reversal, under which the bilinears can be shown to transform into $(+,-,+,+,-)(-)^S$ times the bilinear with $t\to-t$. Note that $\bar\psi\gamma^\mu\psi$ transforms as expected for a current and $\bar\psi\sigma^{kl}\psi$ as expected for an angular momentum. It is only the first of (10.10) that violates time reversal: it is as tricky to violate as CP. In fact there is a deep connection between T and CP known as the CPT theorem.

⋆ The simplest version of QCD, the strong interaction sector of the standard model, violates the chiral symmetry used to rotate away $\bar\psi i\gamma^5\psi$. Then one could get CP violation with a smaller number of generations. To be compatible with the experimental size of CP violation the coefficient of such a term would have to be so tiny that a modified form of QCD which restores this symmetry (and predicts axions) is usually postulated. Then one is back to the three generation requirement.

Composing the three discrete symmetries we find

$$(CPT)^{-1}\psi(x)CPT = e^{-i\tau}(i\gamma^2\gamma^0\gamma^1\gamma^3)\psi(-x) = -e^{-i\tau}\gamma_5(\psi^\dagger)^T(-x).$$

Now applying this transformation to the bilinears, remembering the antilinear property of CPT, we find

$$\begin{aligned}(CPT)^{-1}\bar{\psi}_A(x)\Gamma\psi_B(x)CPT =&\psi_A^T(-x)\gamma_5\beta\Gamma^*\gamma_5(\psi_B^\dagger)^T(-x)\\ =&-\bar{\psi}_B(-x)\beta\gamma_5\Gamma^\dagger\beta\gamma_5\psi_A(-x)\\ =&(-)^{n_\Gamma}\bar{\psi}_B(-x)\beta\Gamma^\dagger\beta\psi_A(-x)\\ =&(-)^{n_\Gamma}(\bar{\psi}_A(-x)\Gamma\psi_B(-x))^\dagger\end{aligned} \tag{10.11}$$

where n_Γ is the number of Lorentz indices carried by Γ. The CPT theorem states the impossibility of violating this symmetry in quantum field theory. We shall not go through the rigorous proof here, but from the transformation law of the bilinears it is clear what is behind the theorem. Since each Lorentz index must be contracted with another in forming a Lorentz scalar polynomial of the bilinears, all of the $(-)^{n_\Gamma}$'s will cancel in the CPT transform of the polynomial. If we denote the Hamiltonian by some function $H(\bar{\psi}_A(x)\Gamma\psi_B(x))$ of the bilinears, we have

$$\begin{aligned}(CPT)^{-1}H(\bar{\psi}_A(x)\Gamma\psi_B(x))CPT =&H^*((CPT)^{-1}\bar{\psi}_A(x)\Gamma\psi_B(x)CPT)\\ =&H^*((\bar{\psi}_A(-x)\Gamma\psi_B(-x))^\dagger)\end{aligned}$$

where by H^* we mean that all of the complex numbers appearing in the formation of H as a function of the bilinears are complex conjugated. Apart from ordering of operators, the last line is just what we mean by the hermitian conjugate of H, if we set $t = 0$ (conservation of energy means H is constant) and integrate over $\mathbf{x}$. So up to operator ordering questions (which can be sorted out for local interactions), a hermitian Hamiltonian must be CPT invariant.

11. Representations of the Poincaré Group for General Spin

We have so far encountered via simple quantum field theories the relativistic quantum description of free particles of spin 0 or spin 1/2. It is useful at this point to realize that much of what we have found is not really tied to field theory but rather to simple requirements of Poincaré invariance. In quantum mechanics it is a general fact that a symmetry group must be realized by a unitary or antiunitary representation. The latter possibility only occurs for some discrete symmetries (time reversal being the physical example). Our goal in this chapter is to obtain the unitary realization of the Poincaré group for multiparticle states.

The Poincaré group consists of Lorentz transformations together with translations

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu.$$

where Λ preserves Minkowski scalar products

$$\eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = \eta_{\mu\nu}.$$

The Λ's can be divided into 4 disjoint sets according to the signs of $\det\Lambda$ and $\Lambda^0{}_0$. This is because it is easy to show from the above property that $(\det\Lambda)^2 = 1$ and $(\Lambda^0{}_0)^2 \geq 1$. Thus a continuous variation of Λ always stays within one of these sets. In the following we restrict ourselves to the *proper* Lorentz Group, *i.e.*with $\det\Lambda = +1$ and $\Lambda^0{}_0 \geq +1$. The complete Lorentz group is then obtained by adjoining parity and time reversal.

Lorentz transformations with $\Lambda^0{}_k = \Lambda^k{}_0 = 0$ are simply rotations and form a subgroup. We know from basic quantum mechanics all the unitary irreducible representations of the Rotation group, namely those labeled by angular momentum $j = 0, 1/2, 1, 3/2, \ldots$. The unitary representations of the Lorentz group must be extensions of these. Let us ask then how to construct this extension for a free massive particle of spin s. Such a particle must be described by a set of at least $2s+1$ momentum space wave functions $f_a(\mathbf{p})$. This much follows just from the Rotation group. We shall find a representation of the Poincaré group with this minimal number of components.

The basic idea, due to Wigner, is to exploit the fact that one can always bring a massive particle to rest by a Lorentz transformation. Define a "standard boost" $B_{\mathbf{p}}$ which boosts a particle at rest to one with momentum $\mathbf{p}$. Let us introduce momentum eigenstates via

$$|f\rangle \equiv \int d^3p \sum_a |\mathbf{p}, a\rangle\, f_a(\mathbf{p}).$$

Then a momentum eigenstate of the particle can be related to the state at rest by

$$|\mathbf{p}, a\rangle \equiv \sqrt{\frac{m}{\omega(\mathbf{p})}} U(B_{\mathbf{p}})\,|\mathbf{0}, a\rangle\,. \tag{11.1}$$

The multiplicative constant is necessary because we want U to be unitary. To understand this point, notice that the relation of the three momentum $\mathbf{p}'$ of a boosted particle to its initial momentum is nonlinear:

$$p'^k = \Lambda^k{}_l p^l + \Lambda^k{}_0 \omega(\mathbf{p}).$$

The Jacobian of this nonlinear transformation of variables is $\partial(\mathbf{p}')/\partial(\mathbf{p}) = \omega(\mathbf{p}')/\omega(\mathbf{p})$. The easiest way to see this is to observe that $\int d^4p\delta(p^2+m^2)$ is a Lorentz invariant; integrating over p^0 then shows that

$\int d^3p/\omega(\mathbf{p})$ is an invariant, which implies the above value for the Jacobian. A general Lorentz transformation on a particle state of momentum $\mathbf{p}$, which boosts it to $\mathbf{p}'$, can be expressed

$$\Lambda = B_{\mathbf{p}'}(B_{\mathbf{p}'}^{-1}\Lambda B_{\mathbf{p}})B_{\mathbf{p}}^{-1}.$$

The transformation in parentheses leaves a particle at rest at rest, and is therefore simply a rotation. Applying $U(\Lambda)$ to (11.1) we have

$$\begin{aligned}
U(\Lambda)\left|\mathbf{p},a\right\rangle =& \sqrt{\frac{m}{\omega(\mathbf{p})}}U(B_{\mathbf{p}'})U(B_{\mathbf{p}'}^{-1}\Lambda B_{\mathbf{p}})\left|\mathbf{0},a\right\rangle \\
=& \sqrt{\frac{m}{\omega(\mathbf{p})}}U(B_{\mathbf{p}'})\left|\mathbf{0},b\right\rangle D^s_{ba}(B_{\mathbf{p}'}^{-1}\Lambda B_{\mathbf{p}}) \\
=& \sqrt{\frac{\omega(\mathbf{p}')}{\omega(\mathbf{p})}}\left|\mathbf{p}',b\right\rangle D^s_{ba}(B_{\mathbf{p}'}^{-1}\Lambda B_{\mathbf{p}}).
\end{aligned}$$

Here $D^s_{ba}(R)$ is just the standard representation matrix of the rotation group with spin s. It is now easily checked that this defines a unitary representation of the Lorentz group on single particle states of spin s. Of course, on momentum eigenstates of a free particle, space-time translations are trivially realized by multiplication of the state by the phase $e^{-ia\cdot p}$ for the space-time translation by amount a^μ.

Let us return to the "standard boost" $B_{\mathbf{p}}$. It is clearly not uniquely determined since it can be preceded by an arbitrary rotation and followed by a rotation about the axis parallel to $\mathbf{p}$. There are two widely used choices for this boost. The simplest choice is the pure boost parallel to $\mathbf{p}$ which we will call $B^0_{\mathbf{p}}$. The second choice is dictated by choosing helicity states. It is described as follows. First agree that the spin states of the particle at rest be labeled by λ the eigenvalue of J_3. Then first boost the particle along the z axis to momentum $|\mathbf{p}|\hat{z}$. Then apply a rotation that carries the z axis to the direction of $\mathbf{p}$. This latter rotation can be taken to be

$$R_0(\mathbf{p}) \equiv e^{-i\phi J_3}e^{-i\theta J_2}e^{+i\phi J_3}$$

where (θ,ϕ) are the polar angles of $\mathbf{p}$. Then the helicity preserving standard boost is given by

$$B^h_{\mathbf{p}} = R_0(\mathbf{p})B^0(|\mathbf{p}|\hat{z}).$$

Then, clearly, $\left|\mathbf{p},\lambda\right\rangle \equiv \sqrt{m/\omega}B^h_{\mathbf{p}}\left|\mathbf{0},\lambda\right\rangle$ is an eigenstate of momentum $\mathbf{p}$ with helicity λ, since rotations do not change helicity. This is easy to see from the general transformation law. For a massive particle, helicity *is* changed by a general Lorentz transformation, *i.e.* $B_{\mathbf{p}'}^{-1}\Lambda B_{\mathbf{p}}$ can be any rotation. However, if one considers the massless limit of this rotation for any fixed Λ, he discovers that it always approaches a rotation about the z axis. Thus for *massless* particles helicity is actually a Lorentz invariant. Thus it can be consistent for massless particles with spin to exist in only one helicity state.

Having understood how a single particle state transforms under Lorentz transformations it is straightforward to find the transformation law for states with any number of free particles of varying mass and spin, which can be viewed as tensor products of single particle states. We can incorporate bose or fermi statistics by introducing a vacuum state $|0\rangle$ and creation and annihilation operators for each species of particle, *e.g.*:

$$\left|\mathbf{p},\lambda\right\rangle \equiv b^{i\dagger}_\lambda(\mathbf{p})\left|0\right\rangle,$$

where i labels the species. When we are dealing with only a limited number of species, we typically choose different letters for different types of particles, *e.g.* $a, a^\dagger$ for neutral scalar particles, $b, b^\dagger$ $(d, d^\dagger)$ for particles (antiparticles), etc. Then the transformation law for a general multiparticle state is completely defined by

$$\begin{aligned}
U(\Lambda)\left|0\right\rangle &= \left|0\right\rangle \\
U(\Lambda)b_\lambda^{i\dagger}(\mathbf{p})U^{-1}(\Lambda) =& b_{\lambda'}^{i\dagger}(\Lambda\mathbf{p})D_{\lambda'\lambda}^{s}(B_{\Lambda\mathbf{p}}^{-1}\Lambda B_{\mathbf{p}}) \\
U(\Lambda)b_\lambda^{i}(\mathbf{p})U^{-1}(\Lambda) =& b_{\lambda'}^{i}(\Lambda\mathbf{p})D_{\lambda'\lambda}^{s*}(B_{\Lambda\mathbf{p}}^{-1}\Lambda B_{\mathbf{p}}).
\end{aligned}$$

Of course the last transformation law is just the Hermitian conjugate of the second one since U is supposed to be unitary. We complete the description by writing down the energy and momentum operators

$$\begin{aligned}
H &= \int d^3p \sum_i \sqrt{\mathbf{p}^2 + m_i^2} \sum_\lambda b_\lambda^{i\dagger}(\mathbf{p})b_\lambda^{i}(\mathbf{p}) \\
\mathbf{P} &= \int d^3p\mathbf{p} \sum_i \sum_\lambda b_\lambda^{i\dagger}(\mathbf{p})b_\lambda^{i}(\mathbf{p}).
\end{aligned}$$

The above discussion might mislead one into thinking that the problem of free quantum field theory for any spin is completely solved. Indeed, we have solved the problem of constructing a relativistic quantum description of any number of free particles with any spins. However, the equivalence of this description to local field theory is not yet transparent. We have explicitly seen how this works for spin 0 and 1/2. which are described by scalar and Dirac fields respectively. The scalar field is supposed to have the Lorentz transformation properties $\phi'(x') = \phi(x)$. We relate this to the general discussion by first identifying the momentum eigenstates with a creation operator applied to the vacuum $\left|\mathbf{p}\right\rangle = a^\dagger(\mathbf{p})\left|0\right\rangle$. Assuming the vacuum is Lorentz invariant, we conclude from the general discussion that

$$U(\Lambda)a^\dagger(\mathbf{p})U^{-1}(\Lambda) = \sqrt{\frac{\omega(\mathbf{p}')}{\omega(\mathbf{p})}}a^\dagger(\mathbf{p}').$$

With this result we can then evaluate how the scalar field transforms

$$\begin{aligned}
U^\dagger(\Lambda)\phi(x)U(\Lambda) =& \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p})}}\sqrt{\frac{\omega(\mathbf{p}')}{\omega(\mathbf{p})}}(a(\mathbf{p}')e^{ix\cdot p} + a^\dagger(\mathbf{p}')e^{-ix\cdot p}) \\
=& \int \frac{d^3p'}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p}')}}(a(\mathbf{p}')e^{ix\cdot\Lambda p'} + a^\dagger(\mathbf{p}')e^{-ix\cdot\Lambda p'}) \\
=& \phi(\Lambda^{-1}x)
\end{aligned}$$

as desired.

Notice that Lorentz covariance alone is achieved by the positive frequency part of the field

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p})}}a(\mathbf{p})e^{ix\cdot p}.$$

But such a field would not commute with its adjoint at space-like separations as a local field must.

$$[\phi^{(+)}(x), \phi^{(-)}(y)] = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega}e^{i(x-y)\cdot p}.$$

Compare this to the result for the total field

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega}(e^{i(x-y)\cdot p} - e^{-i(x-y)\cdot p}).$$

If $(x-y)^2 > 0$ there is a Lorentz frame for which $x^0 = y^0$. In that frame the r.h.s. is manifestly zero. It must be zero in all frames by Lorentz covariance. Notice also that the all important minus sign on the r.h.s came because we chose commutation relations for $a, a^\dagger$. Had we tried to make the scalar particles fermions by imposing anticommutation relations, the two terms would add and locality would be lost. This is the famous spin-statistics connection for the scalar field. For the Dirac field fermi statistics was necessary for stability rather than locality, but the spin-statistics connection is nonetheless fixed.

The Dirac field shows us how we must generalize these considerations to develop a field theory for particles with spin. As shown in an exercise the Dirac field transforms under Lorentz transformations as

$$\psi'(x') = e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/2}\psi(x),$$

where $(e^{-\lambda})^\mu_\nu = \Lambda^\mu_\nu$. The 4×4 matrices $\sigma^{\mu\nu}$ provide a finite dimensional representation of the Lorentz group which is necessarily *not* unitary. This nonunitarity is associated with the noncompactness of the Lorentz group. The nonunitarity of these matrices does not conflict with the unitarity of the action of the Lorentz group on the state space which is just that on multi-particle states we have just discussed. By expressing the field in terms of creation and annihilation operators we can see how the unitary representation on particle states induces the desired field transformation.

From the transformation properties of a spin 1/2 particle it follows that

$$\begin{aligned}
U^\dagger(\Lambda)d^\dagger_\lambda(\mathbf{p})U(\Lambda) =& \sqrt{\frac{\omega(\mathbf{p}')}{\omega(\mathbf{p})}}d^\dagger_{\lambda'}(\mathbf{p}')D^{1/2}_{\lambda'\lambda}(B^{-1}_{\mathbf{p}'}\Lambda^{-1}B_{\mathbf{p}})\\
U^\dagger(\Lambda)b_\lambda(\mathbf{p})U(\Lambda) =& \sqrt{\frac{\omega(\mathbf{p}')}{\omega(\mathbf{p})}}b_{\lambda'}(\mathbf{p}')D^{1/2*}_{\lambda'\lambda}(B^{-1}_{\mathbf{p}'}\Lambda^{-1}B_{\mathbf{p}}).
\end{aligned}$$

where $p' = \Lambda^{-1}p$. Focus on the way b enters the Dirac field:

$$\begin{aligned}
U^\dagger(\Lambda)\int &\frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p})}}\sum_\lambda b_\lambda(\mathbf{p})u_\lambda(\mathbf{p})e^{ix\cdot p}U(\Lambda)\\
=&\int\frac{d^3p'}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p}')}}b_{\lambda'}(\mathbf{p}')u_\lambda(\Lambda\mathbf{p}')e^{i\Lambda^{-1}x\cdot p'}D^{1/2*}_{\lambda'\lambda}(B^{-1}_{\mathbf{p}'}\Lambda^{-1}B_{\mathbf{p}})\\
=&\int\frac{d^3p'}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p}')}}b_{\lambda'}(\mathbf{p}')u_\lambda(\Lambda\mathbf{p}')e^{i\Lambda^{-1}x\cdot p'}D^{1/2}_{\lambda\lambda'}(B^{-1}_{\mathbf{p}}\Lambda B_{\mathbf{p}'})\\
=&e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/2}\int\frac{d^3p'}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p}')}}\sum_{\lambda'}b_{\lambda'}(\mathbf{p}')u_{\lambda'}(\mathbf{p}')e^{i\Lambda^{-1}x\cdot p'},
\end{aligned}$$

which is exactly the desired field transformation. The term involving $d^\dagger$ works in an exactly similar way. In obtaining the last line we used the identity

$$e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/2}u_\lambda(\mathbf{p}) = \sum_{\lambda'}u_{\lambda'}(\Lambda\mathbf{p})D^{1/2}_{\lambda'\lambda}(B^{-1}_{\Lambda\mathbf{p}}\Lambda B_{\mathbf{p}}),$$

which is a simple consequence of the way Lorentz covariance works in the first quantized interpretation of the Dirac wave function.

Clearly the first step in generalizing to higher spin fields is to classify all of the finite dimensional representations of the Lorentz group. We would like to find the possible representation matrices so that a multi-component field $\psi_\alpha(x)$ will have the transformation law

$$\psi'_\alpha(x) = D_{\alpha\beta}(\Lambda)\psi_\beta(\Lambda^{-1}x).$$

Let us first cast the algebra of Lorentz generators

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho})$$

in terms of the generators for rotations $J_k = \epsilon_{klm}M_{lm}/2$ and for boosts $K_k = M_{0k}$:

$$\begin{aligned}
[J_k, J_l] =& i\epsilon_{klm}J_m \\
[J_k, K_l] =& i\epsilon_{klm}K_m \\
[K_k, K_l] =& -i\epsilon_{klm}J_m.
\end{aligned}$$

Now notice that the linear combinations $\mathbf{J}_\pm \equiv (\mathbf{J} \pm i\mathbf{K})/2$ satisfy the algebra of two mutually commuting angular momentum algebras

$$\begin{aligned}
[J_k^\pm, J_l^\pm] =& i\epsilon_{klm}J_m^\pm \\
[J_k^+, J_l^-] =& 0.
\end{aligned}$$

We know from elementary quantum mechanics what all of the finite dimensional representations of the rotation group are: they are labeled by the eigenvalues $j(j+1)$ of the Casimir operator $\sum_k J^{k2}$ with $j = 0, 1/2, 1, 2, \ldots$ all nonnegative integers and half integers. The representation j has dimension $2j+1$. It follows that all the finite dimensional representations of the Lorentz group are labeled by the pair of eigenvalues $j_+(j_+ + 1), j_-(j_- + 1)$ of the pair of Casimir operators $\sum_k J_\pm^{k2}$ where $2j_+, 2j_-$ are any pair of nonnegative integers. All of these representations are equivalent to a unitary representation, *i.e.* $\mathbf{J}_\pm$ are both represented by hermitian matrices. This means that the rotation generators $\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-$ are represented by hermitian matrices, but the boost generators $\mathbf{K} = -i(\mathbf{J}_+ - \mathbf{J}_-)$ are represented by antihermitian matrices. Thus the finite dimensional representations of the Lorentz group are not equivalent to unitary ones. This is associated with the fact that the Lorentz group is noncompact. We already encountered this nonunitarity in the representation of the Lorentz group by gamma matrices.

We denote the representation matrices by $D(j_+, j_-)$. The simplest nontrivial representations are $D(1/2, 0)$ and $D(0, 1/2)$. In the first the generators J_- are represented by 0 and the generators J_+ by $\sigma/2$. This means that the angular momentum is represented by $\mathbf{J} = \sigma/2$ and the boost generators by $\mathbf{K} = -i\sigma/2$. The other two dimensional representation has the same representative for $\mathbf{J}$ but the boost is represented by $\mathbf{K} = +i\sigma/2$. It is clear that these representations are not equivalent to each other, since any similarity transformation which could reverse the sign of $\mathbf{K}$ would do the same to $\mathbf{J}$. However these two inequivalent representations can be related by complex conjugation. In fact it is easy to see from the properties of the Pauli matrices that $D(1/2, 0)^* = \sigma_2 D(0, 1/2)\sigma_2$. In general, the representation $D(k, m)^*$ is equivalent to $D(m, k)^\star$, so the only real irreducible representations have $j_+ = j_-$. Of course $D(k, m) \oplus D(m, k)$ is real but it is also reducible.

Notice that parity reverses the sign of the boost generators but not the sign of the angular momentum. Thus the representations $D(k, m)$ and $D(m, k)$ are also related by parity. We encountered this fact with the Dirac field which admits the parity symmetry. It exploits the reducible representation $D(1/2, 0) \oplus D(0, 1/2)$

⋆ In $D(k, m)^*$ the generators are $-\mathbf{J}^*, -\mathbf{K}^*$ so *e.g.* J_+ is represented by $-J^* - iK^* = -(J_-)^*$ and *vice versa*.

to achieve this. To make this quite explicit we note that the representatives of the Lorentz generators are the 4×4 matrices

$$\mathbf{J} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$
$$\mathbf{K} = \frac{-i}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}$$

which are just the components of $\sigma^{\mu\nu}/2$ constructed out of gamma matrices in the so-called natural representation

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$
$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}.$$

Defining $\sigma^\mu = (I, \boldsymbol{\sigma})$, and $\bar{\sigma}^\mu = (I, -\boldsymbol{\sigma})$ we can unify these two in the single equation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

The potential spin content of a field in a given representation is typically richer than one might desire. For example, since $J = J_+ + J_-$ the representations of the rotation group contained in $D(k, m)$ include all spins that arise from adding spin k to spin m: $|k-m|, |k-m|+1, \ldots, k+m$. Thus if our desire is to describe a given spin, depending on our choice of representation, we might bring in several other spins as well. The choice of field content is not unambiguous. Even for spin $1/2$ we have noted various possibilities, *e.g.* Dirac, Majorana, and Weyl. Weyl fermions make use of $D(1/2, 0)$, but since this is not a real representation, the hermitian conjugate field, which transforms under $D(0, 1/2)$, must also be introduced and represents the anti-particle.

When we come to spin 1, two possibilities come to mind. $D(1,0) \oplus D(0,1)$ or $D(1/2, 1/2)$. The latter contains potentially both spin 1 and spin 0 and is in fact the representation of a four-vector field. The former seems to contain spin 1 twice. It is easy to see that it is the antisymmetric tensor product of $D(1/2, 1/2)$ with itself and thus represents an antisymmetric second rank tensor. The field strengths $F_{\mu\nu}$ of electromagnetism spring to mind, so we might decide that the first choice is best. However, we know that in quantum mechanics it is necessary to use the potentials A_μ which transform under the second choice. Then gauge invariance is essential to eliminate unwanted spin states.

This general discussion of higher spin serves to indicate some of the subtleties and complexities that must be confronted. In fact, consistent fully interacting quantum field theories have never been constructed for spins higher than 2 (the graviton). Furthermore, ultraviolet divergences have so far caused incurable difficulties for theories with spin higher than 1 including quantum gravity. Since gravity is very much present in the real world, it is clear that there is much to do before we can claim that quantum field theory can describe all of physics.

12. Quantum Field Equations with External Fields

12.1. Electromagnetic Fields

The coupling of the Dirac equation to an external electromagnetic field is dictated by the principle of gauge invariance. In classical electrodynamics, it is possible to avoid potentials and formulate all equations of motion in terms of the electric and magnetic fields $F_{\mu\nu}$. However, the potential $A_\mu(x)$ is indispensable to an economical description of the coupling of a quantum particle to electromagnetism. Fundamentally, this is because the Hamiltonian and Lagrangian play a much more central role in quantum dynamics than in classical dynamics, and the potential appears explicitly in the Hamiltonian and Lagrangian. (Recall that the Schrödinger equation involves the Hamiltonian explicitly.) The field strengths are related to the potential via

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x),$$

but it is clear that the potential is not given uniquely in terms of the field strength: If the potential is changed by a *gauge transformation*

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \tag{12.1}$$

the field strength $F_{\mu\nu}$ is unchanged. It is therefore important to introduce the potential into the Schrödinger equation in a way which preserves gauge invariance.

Let us first ask how gauge invariance is realized in classical particle electrodynamics. In order that the Euler-Lagrange equations reproduce the Lorentz force law, the scalar and vector potentials $(A^0, \mathbf{A})$ must enter the Lagrangian through the terms

$$-qA^0(x) + q\dot{\mathbf{x}} \cdot \mathbf{A}(x).$$

Because of the term linear in velocity, the momentum conjugate to $\mathbf{x}$ becomes

$$\mathbf{p} = \mathbf{p}_{\mathbf{A}=0} + q\mathbf{A},$$

where $\mathbf{p}_{\mathbf{A}=0}$ is the conjugate momentum with vanishing vector potential. Furthermore, when we form the Hamiltonian $H = \dot{\mathbf{x}} \cdot \mathbf{p} - L$, the term linear in velocity cancels, so

$$H = H_0(\mathbf{p}_{\mathbf{A}=0}, \mathbf{x}) + qA^0 = H_0(\mathbf{p} - q\mathbf{A}, \mathbf{x}) + qA^0.$$

If we subject A to a gauge transformation, the Lagrangian changes by the amount

$$q(\partial_0 \Lambda + \dot{\mathbf{x}} \cdot \nabla \Lambda) = q\frac{d\Lambda}{dt},$$

a total time derivative, so that the action $\int_{t_1}^{t_2} L$ changes by the amount $q\Lambda(\mathbf{x}(t_2), t_2) - q\Lambda(\mathbf{x}(t_1), t_1)$ and the added terms have no effect on the Euler-Lagrange equations. At first sight the Hamiltonian doesn't look invariant, but notice that the transformation

$$\mathbf{p}' = \mathbf{p} - q\nabla\Lambda \qquad\qquad \mathbf{x}' = \mathbf{x}$$

is a canonical transformation with generating function $W_2(\mathbf{x}, \mathbf{p}', t) = \mathbf{x} \cdot \mathbf{p}' + q\Lambda(\mathbf{x}, t)$. Furthermore if the generating function is time dependent the canonical transformation includes changing the Hamiltonian by $\partial_t W_2 = q\partial_t \Lambda$, so that after the canonical transformation the gauge transformed Hamiltonian is identical to the old one with the substitutions $\mathbf{x} \to \mathbf{x}'$, $\mathbf{p} \to \mathbf{p}'$.

The way A_μ enters the Schrodinger equation for a charged particle is now clear. The substitution $\mathbf{p} \to \mathbf{p} - q\mathbf{A}$ corresponds in the Schrödinger equation to

$$\frac{1}{i}\nabla \to \frac{1}{i}\nabla - q\mathbf{A}.$$

Furthermore the addition of $qA^0 = -qA_0$ to the Hamiltonian is prescribed by the substitution rule

$$\frac{1}{i}\frac{\partial}{\partial t} \to \frac{1}{i}\frac{\partial}{\partial t} - qA_0,$$

so both substitutions can be given the compact expression

$$\partial_\mu \to \partial_\mu - iqA_\mu \tag{12.2}$$

which is known as the minimal substitution rule. The gauge invariance of the Schrödinger equation is achieved by postulating in addition to (12.1) the change

$$\psi \to e^{iq\Lambda}\psi. \tag{12.3}$$

If we recall that in the semi-classical approximation $\psi \sim e^{iS}$, where S is the classical action, we recognize that this change of the wave function under gauge transformation is the quantum analogue of the classical change in the action.

The gauge transformation makes an arbitrary, local, redefinition of the phase of the wave function. In fact, one could take the attitude that invariance under such local phase changes is desirable from a physical point of view. (Since global phase changes are unobservable perhaps local ones should also be.) In that case one would be forced to introduce the electromagnetic field to realize the invariance! It is obvious that the Schrödinger equation is invariant under the combined changes (12.1) and (12.3). As a special case, the Dirac equation is invariant under the same gauge transformations. When we interpret the Dirac equation as a field equation, (12.3) is a transformation on fields as is (12.1), so the two are really on similar footing.

To sum up the above discussion, we display the Dirac equation in a potential A_μ:

$$i\gamma \cdot (\partial - iqA)\psi = m\psi.$$

Also the corresponding second quantized Hamiltonian is given by

$$H_A(t) = \int d^3x(\psi^\dagger(\frac{1}{i}\boldsymbol{\alpha} \cdot \nabla + \beta m)\psi - j^\mu A_\mu).$$

Notice that a simple consequence of the Dirac equation is current conservation

$$\partial_\mu j^\mu = 0,$$

even when $A \neq 0$. This can be understood as a consequence of the gauge invariance of quantum evolution. To see this we have to consider the unitary evolution operator $U(t, t_0)^\star$ defined by

$$i\partial_t U(t, t_0) = H_S(t)U(t, t_0) \qquad\qquad U(t_0, t_0) = I, \tag{}$$

where H_S is the Schrödinger picture Hamiltonian. If we make a small change δA_μ in A_μ, **keeping the**

$\star$ U gives the unitary transformation between Heisenberg and Schrödinger pictures. Its analogue in classical mechanics is the generator $S(q, P, t)$ of the canonical transformation mapping the initial phase space variables Q, P to those at time t, q, p. The analogue of the following equation for U is the Hamilton-Jacobi equation for S,

$$\frac{\partial S}{\partial t} = -H(q, \frac{\partial S}{\partial q}, t).$$

Schrödinger picture dynamical variables fixed, U changes by the amount[†]

$$\begin{aligned}\delta U(t,t_0) =& i\int_{t_0}^{t} dt' d^3x U(t,t')\delta A_\mu(x')j^\mu(\mathbf{x})U(t',t_0)\\ =& U(t,t_0)i\int_{t_0}^{t} dt' d^3x\delta A_\mu(x')j^\mu(\mathbf{x},t')\end{aligned} \tag{12.4}$$

with $j^\mu(\mathbf{x},t') = U^\dagger(t',t_0)j^\mu(\mathbf{x})U(t',t_0)$ the Heisenberg picture current operator[‡]. To obtain this result simply differentiate (12.4) with respect to time and show that it satisfies () to first order in δA. Use is also made of the closure relation

$$U(t,t')U(t',t_0) = U(t,t_0)$$

which is a simple consequence of the differential equation satisfied by U. Under a gauge transformation $\delta A = \partial\Lambda$ so we see that

$$\delta U = Ui\int_{t_0}^{t} d^4x\partial_\mu\Lambda j^\mu(x).$$

The requirement that U be invariant under gauge transformations that vanish at t_0, t is then that $\partial_\mu j^\mu = 0$.

12.2. Nonabelian Gauge Fields

The local phase transformation (12.3) on charged fields can be generalized (Yang-Mills). Suppose that ψ carries an internal index k. Then in place of (12.3) we can consider

$$\psi_k(x) \to \sum_l \Omega_{kl}(x)\psi_l(x)$$

or, with suppressed indices

$$\psi(x) \to \Omega(x)\psi(x) \tag{12.5}$$

with $\Omega(x)$ in a unitary matrix representation of some continuous group G. In this language (12.3) corresponds to the choice $G = U(1)$, multiplication by a phase, an abelian group. If we require the dynamics to be

† In the classical theory a change in the parameters of the Hamiltonian would give rise to a change in S satisfying

$$\frac{\partial\delta S}{\partial t} = -\sum_k \frac{\partial\delta S}{\partial q_k}\frac{\partial H}{\partial p_k} - \delta H(q, \frac{\partial S}{\partial q}, t).$$

Now if we put $q = q(t)$ and $\partial S/\partial q = p(t)$ the Hamilton-Jacobi equation just says that $q(t), p(t)$ satisfy Hamilton's equations so $\partial H/\partial p = \dot{q}$ and we have

$$\frac{d\delta S}{dt} = -\delta H(q(t), p(t), t)$$

or $\delta S = -\int_{t_0}^t dt'\delta H(q(t'), p(t'), t')$, the classical analogue of the following equation.

‡ Equation (12.4) has a generalization to an arbitrary physical system: If one makes any small change δH_S in the Schrödinger picture hamiltonian, with the Scrödinger picture operators fixed, the corresponding change in the evolution operator $U(t,t_0)$ is given by $\delta U = -iU\int_{t_0}^t dt'\delta H(t')$, where $\delta H(t) \equiv U^\dagger \delta H_S U$ is the change in the Schrödinger hamiltonian transformed to the Heisenberg picture, which is the same as the change in the Heisenberg Hamiltonian $H(t)$, keeping the Heisenberg dynamical variables fixed. The proof of this is exactly the same as that of Eq.(12.4).

invariant under the nonabelian local gauge transformations, we must introduce a nonabelian gauge field to absorb the noncovariance of

$$\partial_\mu \psi \to \Omega(\partial_\mu + \Omega^{-1}\partial_\mu \Omega)\psi.$$

In analogy with the electromagnetic case, we need to introduce a matrix valued potential $A_\mu(x)$ via the substitution

$$\partial_\mu \to \partial_\mu - igA_\mu(x) \equiv D_\mu. \tag{12.6}$$

Then the requirement $D_\mu \psi \to \Omega D_\mu \psi$ translates to

$$A_\mu \to \Omega A_\mu \Omega^{-1} - \frac{i}{g}\partial_\mu \Omega \Omega^{-1}. \qquad (12.7).$$

Clearly A takes values in the Lie algebra of the group G. If ψ is a Dirac field, a gauge invariant dynamics is given by the field equation

$$i\gamma \cdot D\psi = m\psi,$$

or the corresponding second quantized Hamiltonian

$$H_A(t) = \int d^3x(\psi^\dagger(\frac{1}{i}\boldsymbol{\alpha}\cdot\nabla + \beta m)\psi - g\bar{\psi}A_\mu\gamma^\mu\psi).$$

Just as in the electromagnetic case we can consider how the quantum evolution operator changes under a small change of A, and identical steps lead to:

$$\delta U(t,t_0) = U(t,t_0)i\int_{t_0}^{t} dt' d^3x \bar{\psi} g \delta A_\mu \gamma^\mu \psi.$$

An infinitesimal gauge transformation $\Omega = I + ig\epsilon G$ corresponds to

$$\delta A_\mu = \epsilon(\partial_\mu G - ig[A_\mu, G]).$$

For G which vanish initially and finally the corresponding change in U is

$$\begin{aligned}\delta U(t,t_0) =& U(t,t_0)ig\epsilon\int_{t_0}^{t} dt' d^3x \\ & G_{ab}(-\partial_\mu(\bar{\psi}_a\gamma^\mu\psi_b) - ig(A_\mu)_{ca}\bar{\psi}_c\gamma^\mu\psi_b + ig(A_\mu)_{bc}\bar{\psi}_a\gamma^\mu\psi_c)\end{aligned}$$

so gauge invariance implies the following generalization of current conservation ("covariant conservation")

$$D_\mu j^\mu \equiv \partial_\mu j^\mu - ig[A_\mu, j^\mu] = 0,$$

where the current is a matrix operator

$$j^\mu(x)_{ba} \equiv \bar{\psi}_a(x)\gamma^\mu\psi_b(x).$$

According to the Principle of Equivalence, an external gravitational field is described by introducing a space-time dependent metric $\eta_{\mu\nu} \to g_{\mu\nu}(x)$ which then must enter the field equations in a generally covariant way. This prescription suffices for bosonic fields but new concepts must be brought in for fermionic fields. At this stage we shall confine our discussion to a real scalar field. The minimal generally covariant classical action is given by

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}(g^{\mu\nu}(x)\partial_\mu\phi\partial_\nu\phi + m^2\phi^2).$$

To construct the Hamiltonian, we first define the conjugate momentum

$$\pi(x) \equiv \frac{\delta S}{\delta\dot{\phi}(x)} = -\sqrt{-g}g^{0\nu}(x)\partial_\nu\phi.$$

Then the Hamiltonian is

$$\begin{aligned}H(t) &= \int d^3x(\pi(x)\dot{\phi}(x) + \frac{1}{2}\sqrt{-g}(g^{\mu\nu}(x)\partial_\mu\phi\partial_\nu\phi + m^2\phi^2)) \\ &= \int d^3x\left(-\frac{1}{2}\frac{\pi^2}{g^{00}\sqrt{-g}} + \frac{g^{0k}}{g^{00}}\pi\partial_k\phi + \frac{1}{2}\sqrt{-g}\left(g^{kl} - \frac{g^{0k}g^{0l}}{g^{00}}\right)\partial_k\phi\partial_l\phi + \frac{m^2}{2}\sqrt{-g}\phi^2\right).\end{aligned}$$

Just as with gauge fields we can ask how the evolution operator changes under a small change $\delta g_{\mu\nu}$ in the metric, $\delta U = -iU\int_{t_0}^t dt'\delta H(t')$, where δH is computed holding ϕ, π fixed. The easiest way to do this is to evaluate the change in the Lagrangian L at fixed $\phi, \dot{\phi}$. Since L is related to H by a Legendre transform, $\delta H = -\delta L$.

$$\delta L = -\frac{1}{2}\int d^3x\sqrt{-g}\delta g^{\mu\nu}(x)T_{\mu\nu}(x)$$

with $T_{\mu\nu}$ the energy momentum tensor

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + m^2\phi^2).$$

Thus we have finally*

$$\delta U = iU\frac{1}{2}\int_{t_0}^{t} d^4x'\sqrt{-g}\delta g_{\mu\nu}(x')T^{\mu\nu}(x').$$

Under the infinitesimal general coordinate transformation $x^\mu = x'^\mu + \xi^\mu(x')$, the metric changes according to

$$\begin{aligned}g_{\mu\nu}(x) =& g'_{\mu\nu}(x') - \frac{\partial\xi^\rho}{\partial x^\mu}g_{\rho\nu} - \frac{\partial\xi^\rho}{\partial x^\nu}g_{\rho\mu} \\ =& g'_{\mu\nu}(x') - D_\mu\xi_\nu - D_\nu\xi_\mu + \Gamma^\rho_{\mu\sigma}\xi^\sigma g_{\rho\nu} + \Gamma^\rho_{\nu\sigma}\xi^\sigma g_{\rho\mu} \\ =& g'_{\mu\nu}(x') - D_\mu\xi_\nu - D_\nu\xi_\mu + \xi^\rho\partial_\rho g_{\mu\nu} \\ =& g'_{\mu\nu}(x) - D_\mu\xi_\nu - D_\nu\xi_\mu\end{aligned}$$

An infinitesimal change of integration variables is just a surface term:

$$\int d^4x'\mathcal{L}'(x') = \int d^4x(1 - \partial_\rho\xi^\rho)(1 - \xi^\rho\partial_\rho)\mathcal{L}'(x) = \int d^4x\mathcal{L}'(x) - \int d^4x\partial_\rho(\xi^\rho\mathcal{L}'(x))$$

so choosing $\delta g_{\mu\nu} = -D_\mu\xi_\nu - D_\nu\xi_\mu$ should give $\delta U = 0$ for all ξ vanishing sufficiently rapidly at infinity, if the quantum field dynamics is invariant under general coordinate transformations. Thus general coordinate

$\star$ Note that since $g^{\mu\nu}$ is the inverse matrix to $g_{\mu\nu}$, $\delta g^{\mu\nu} = -g^{\mu\rho}\delta g_{\rho\sigma}g^{\sigma\nu}$.

invariance implies that the energy momentum tensor is covariantly conserved:

$$D_\mu T^{\mu\nu} = 0.$$

In the limit of flat space (no gravity) this condition reduces to ordinary energy-momentum conservation.

12.4. Asymptotic States and Matrix Elements

In discussing time dependent processes, it is convenient to introduce asymptotic states which are eigenstates of $H(\pm\infty)$. We denote by $|in\rangle$ the ground state of $H(-\infty)$ and by $|out\rangle$ the ground state of $H(+\infty)$. The normal situation will be one in which all external fields vanish at sufficiently early and late times. Thus $|in\rangle$ and $|out\rangle$ will typically be ground states of $H_0(-\infty)$ and $H_0(+\infty)$ respectively. Although these operators are not the same (because their time evolution is governed by H not H_0), the spectra of the two Hamiltonians *are* identical: $H_0(t) = U^{-1}(t,-\infty)H_{0S}U(t,-\infty)$. By convention we are identifying the Schrödinger and Heisenberg pictures at $t = -\infty$. Thus, if $|in\rangle$ is the ground state of $H_0(-\infty) = H_{0S}$, the state $\langle in|\,U(\infty,-\infty)$ is an eigenstate of $H_0(+\infty)$ with the same eigenvalue and hence the ground state. Thus we can and shall fix phases by defining

$$\langle out| \equiv \langle in|\,U(\infty,-\infty).$$

We stress that this is the true "out" state only when $H_S(\infty) = H_S(-\infty) \equiv H_0$.

If the time dependence of H_S is *adiabatic*, *i.e.* very slow on the time scale set by the level spacings, the Adiabatic Theorem assures us that an eigenstate of $H_S(-\infty)$ evolves to an eigenstate of $H_S(t)$ for all t for which adiabatic conditions apply, even after a long enough time to change H_S by a finite amount. For example, the state $|in\rangle$ will be an eigenstate of $H(t)$ for all t for which adiabatic time variation applies. In particular, the ground state eigenvalue $E_G(t)$ must not get close to the next higher eigenvalue as t varies. If this situation holds for all time, it follows that the state $|in\rangle$ is a phase times the state $|out\rangle$, or $\langle out|$ is this same phase times $\langle in|$. This phase is easily evaluated in terms of the time dependent ground state energy $E_G(t)$ of $H_S(t)$ by applying the Schrödinger equation to $\langle in|\,U(t,-\infty)\,|in\rangle$ and using the adiabatic theorem $H_S(t)U(t,-\infty)\,|in\rangle = E_G(t)U(t,-\infty)\,|in\rangle$:

$$\langle out|in\rangle = \exp\left\{-i\int_{-\infty}^{\infty} dt E_G(t)\right\} \qquad \text{Adiabatic Conditions.}$$

Note carefully that adiabatic conditions would *not* apply if the ground state energy got close to an excited level as time evolved. In particular, it would not apply in processes with pair production when $|\,\langle out|in\rangle\,| < 1$.

It is only for very particular external fields that one can solve the Dirac equation or any field equation exactly. Important examples include the static Coulomb potential for which one can find all the energy eigenvalues and eigenstates, arbitrary constant field strengths, and plane waves. The Coulomb potential is of special importance since the exact solution in that case is the starting point for the relativistic theory of atomic energy levels. Still the complete dynamics is never exactly given by these special cases and perturbation theory is the important tool for evaluating corrections to the exactly soluble (idealized) case, which can be zero external fields or one of the above cases. When we quantize the electromagnetic field perturbation theory is essentially our *only* tool for computing radiative corrections due to the quantum nature of the electromagnetic field.