Quantum Field Theory

Solution Set 1

Due: 18 January 2019

Reading: S, Sections 1-3; Lecture Notes, Chapter 1.

This and all future homework will be posted on the course webpage:

http://www.phys.ufl.edu/∼thorn/homepage/qftinfo.html

In the homework assignments, I will refer to problems in Srednicki's book by prefixing the problem number with S. Note that there are many problems in his book which ask the reader to fill in gaps in the preceding discussion (e.g. S 2.1 through 2.7). I will generally not assign such problems to be graded, but you should nonetheless make sure you understand how to do them.

1. **Lorentz Invariance**

a) Show by direct calculation that the Lorentz boost in the 1 direction leaves the Minkowski scalar product invariant.

Solution: The boost transforms any four vector as $v^{0\prime} = \gamma(v^0 + Vv^1)$; $v^{1\prime} = \gamma(v^1 + Vv^0)$. Applying this transformation to v and w, we calculate

$$\begin{aligned} v' \cdot w' &= \gamma^2(v^1 + Vv^0)(w^1 + Vw^0) + v^2w^2 + v^3w^3 - \gamma^2(v^0 + Vv^1)(w^0 + Vw^1) \\ &= \gamma^2[v^1w^1(1 - V^2) - v^0w^0(1 - V^2)] + v^2w^2 + v^3w^3 = v \cdot w \end{aligned} \tag{1}$$

because $\gamma^2 = 1/(1 - V^2)$.

b) For a general Lorentz transformation, specified by the matrix $\Lambda^\mu{}_\nu$, prove that $(\Lambda^0_0)^2 \geq 1$.

Solution: If Λ is a Lorentz transformaion it satisfies

$$\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \tag{2}$$

or on matrix language $\Lambda^T\eta\Lambda = \eta$. Taking the deteminant of both sides of this equation gives $\det\eta \det\Lambda^T \det\Lambda = \det\eta$ or $(\det\Lambda)^2 = 1$. Moreover setting $\rho = \sigma = 0$ the defining equation reads

$$-(\Lambda^0_0)^2 + \sum_k (\Lambda^k_0)^2 = -1 \qquad \text{or} \qquad (\Lambda^0_0)^2 = 1 + \sum_k (\Lambda^k_0)^2 \geq 1 \tag{3}$$

2. **Lorentz commutator algebra** Consider the $\boldsymbol{x}$ and $\boldsymbol{p}$ operators for a quantum particle. The generators of rotations are the components of angular momentum $\boldsymbol{J} = \boldsymbol{x} \times \boldsymbol{p}$. In class we constructed the Lorentz boost generators

$$\boldsymbol{K} = -(\boldsymbol{x}\sqrt{\boldsymbol{p}^2 + m^2} + \sqrt{\boldsymbol{p}^2 + m^2}\boldsymbol{x})/2 + \boldsymbol{p}t.$$

The Lorentz algebra is:

$$[J^k, J^l] = i\hbar\epsilon_{klm}J^m, \qquad [K^k, J^l] = i\hbar\epsilon_{klm}K^m, \qquad [K^k, K^l] = -i\hbar\epsilon_{klm}J^m.$$

The first commutator is known from basic quantum mechanics, and the second one is a consequence of the fact that $\boldsymbol{K}$ is a vector operator. Using the canonical commutations relations $[x^k, p^l] = i\hbar\delta_{kl}$, prove the third commutator, which is new. Also transcribe these commutators to covariant notation using $M_{ij} = \epsilon_{ijk}J^k$ and $M_{0i} = K^i$ to form the tensor $M_{\mu\nu}$ confirming Eq. (1.17) in the Lecture Notes.

Solution: Since $\boldsymbol{J}$ generates rotations, it commutes with any scalar operator so $[\boldsymbol{J}, \sqrt{\boldsymbol{p}^2+m^2}] = 0$. Also $[x^k, J^l] = i\epsilon_{klm}x^m$, so it immediately follows that $[K^k, J^l] = i\epsilon_{klm}K^m$. The JJ commutator follows from the interpretation of $\boldsymbol{J}$ as generator of rotations. So we only need calculate $[K^k, K^l]$. First calculate

$$[x^k, \sqrt{\boldsymbol{p}^2+m^2}] = i\nabla_k\sqrt{\boldsymbol{p}^2+m^2} = i\frac{p^k}{\sqrt{\boldsymbol{p}^2+m^2}} \tag{4}$$

with which we can rewrite $K^k = \omega(\boldsymbol{p})x^k + ip^k/(2\omega(\boldsymbol{p}))$ Then with $\omega \equiv \sqrt{\boldsymbol{p}^2+m^2}$

$$\begin{aligned}[K^k, K^l] &= [\omega x^k, \omega x^l] + \left[\omega x^k, \frac{ip^l}{2\omega}\right] - \left[\omega x^l, \frac{ip^k}{2\omega}\right] \\ &= \omega[x^k, \omega]x^l] + \omega[\omega, x^l]x^k \\ &= i(p^k x^l - p^l x^k) = -i\epsilon_{klm}J^m \end{aligned} \tag{5}$$

The last two terms on the right of the first line cancel because the commutators either give a factor δ_{kl} or a factor $p^k p^l$ both of which are symmetric under $k \leftrightarrow l$. Finally we evaluate

$$\begin{aligned}[M^{jk}, M^{lm}] &= \epsilon_{jkn}\epsilon_{lmp}[J^n, J^p] = \epsilon_{jkn}\epsilon_{lmp}i\epsilon_{npq}J^q = \epsilon_{jkn}\epsilon_{lmp}iM^{np} \\ &= i(\delta_{jl}M^{km} - \delta_{kl}M^{jm} - \delta_{jm}M^{kl} + \delta_{km}M^{jl}) \end{aligned} \tag{6}$$

$$\begin{aligned}[M^{jk}, M^{l0}] &= \epsilon_{jkn}[J^n, K^l] = i\epsilon_{jkn}\epsilon_{nlm}K^m \\ &= i(\delta_{jl}\delta_{nm} - \delta_{jm}\delta_{kl})K^m = i(\delta_{jl}M^{n0} - \delta_{kl}M^{j0} \end{aligned} \tag{7}$$

$$[M^{j0}, M^{k0}] = [K^j, K^k] = -i\epsilon_{jkm}J^m = -iM^{jk} = i\eta_{00}M^{jk} \tag{8}$$

or in covariant notation

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\sigma}M^{\mu\rho}) \tag{9}$$

3. S, Problem 2.9, part d) only.

Solution:

a)$(I+\delta\omega)^\mu{}_\rho\partial^\rho\phi(x-\delta\omega x) \sim \partial^\mu\phi(x) - \delta\omega^\lambda{}_\sigma x^\sigma\partial_\lambda\partial^\mu\phi(x) + \delta\omega^\mu{}_\rho\partial^\rho\phi(x)$. Thus

$$\frac{i\delta\omega^{\kappa\nu}}{2}[\partial^\mu\phi, M_{\kappa\nu}] = -\delta\omega^\lambda{}_\sigma x^\sigma\partial_\lambda\partial^\mu\phi(x) + \delta\omega^\mu{}_\rho\partial^\rho\phi(x) \tag{10}$$

$$= \frac{i\delta\omega^{\lambda\sigma}}{2}\mathcal{L}_{\lambda\sigma}\partial^\mu\phi(x) + \frac{i\delta\omega^{\lambda\sigma}}{2}(S_{V\lambda\sigma})^\mu{}_\rho\partial^\rho\phi(x) \tag{11}$$

and result follows by comparing coefficients of $\delta\omega$.

b)Following steps to the result of 2.8(d), leads to

$$[\partial^\kappa\phi,[M^{\mu\nu},M^{\rho\sigma}]]=\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\partial^\kappa\phi-\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\partial^\kappa\phi+[S_V^{\mu\nu},S_V^{\rho\sigma}]^\kappa{}_\eta\partial^\eta\phi \tag{12}$$

Plugging in commutator of M's show that commutator of S_V's must match.

c)The 12 block of $(-iS_V^{12})^n$ is $\begin{pmatrix}0 & -1\\ +1 & 0\end{pmatrix}^n$ which is $(-)^{n/2}I$ if n is even and $(-)^{(n-1)/2}\begin{pmatrix}0 & -1\\ +1 & 0\end{pmatrix}$ if n is odd. Then in the 12 subblock:

$$\left[\sum_{n=0}^{\infty}\frac{1}{n!}(-i\theta S_V^{12})^n\right]_{12\text{Block}}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \tag{13}$$

d) The 03 block of $(iS_V^{30})^n$ is $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^n$ which is I if n is even and $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ if n is odd. Then in the 03 subblock:

$$\left[\sum_{n=0}^{\infty}\frac{1}{n!}(i\eta S_V^{30})^n\right]_{03\text{Block}}=\begin{pmatrix}\cosh\eta & \sinh\eta\\ \sinh\eta & \cosh\eta\end{pmatrix} \tag{14}$$

4. **The Scalar Field**

a)Show that the Klein-Gordon scalar wave equation is invariant under a Lorentz trasformation if the field ϕ transforms as a scalar field, *i.e.* $\phi'(x')=\phi(x)$. Here $x'^\mu=\Lambda^\mu_\nu x^\nu$ is a Lorentz transformation.

Solution: $\phi'(x')=\phi(\Lambda^{-1}x')$ so we have

$$\partial'_\mu\partial'_\nu\phi'=(\Lambda^{-1})_\mu{}^\rho(\Lambda^{-1})_\nu{}^\sigma\partial_\rho\partial_\sigma\phi$$

Since Λ^{-1} is a Lorentz transformation, $\eta^{\mu\nu}(\Lambda^{-1})_\mu{}^\rho(\Lambda^{-1})_\nu{}^\sigma=\eta^{\rho\sigma}$, so it follows that

$$\partial'^2\phi'=\eta^{\mu\nu}\partial'_\mu\partial'_\nu\phi'=\partial^2\phi'=m^2\phi'$$

as desired.

b) Show by direct substitution of the continuum field expansions

$$\phi(\boldsymbol{x}) = \int\frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\hbar}{2\omega(\boldsymbol{k})}}(a(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}+a^\dagger(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}) \tag{15}$$

$$\pi(\boldsymbol{x}) = -i\int\frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\hbar\omega(\boldsymbol{k})}{2}}(a(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}-a^\dagger(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}) \tag{16}$$

into the formulas for the energy and momentum of the continuum scalar field that

$$H_\phi - E_0 = \int d^3k\hbar\omega(\boldsymbol{k})a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) \tag{17}$$

$$\boldsymbol{P} = -\int d^3x\pi(\boldsymbol{x})\nabla\phi(\boldsymbol{x}) = \int d^3k\hbar\boldsymbol{k}a^\dagger(\boldsymbol{k})a(\boldsymbol{k}). \tag{18}$$

Solution: When ϕ, Π are plugged into $H = \frac{1}{2}\int d^3x[\Pi^2 + (\nabla\phi)^2 + m^2\phi^2]$, the $\int d^3x$ produces delta functions:

$$\int d^3x\Pi^2 = \int d^3k\frac{\hbar\omega}{2}[a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) + a(\boldsymbol{k})a^\dagger(\boldsymbol{k}) - a(\boldsymbol{k})a(-\boldsymbol{k}) - a^\dagger(\boldsymbol{k})a^\dagger(-\boldsymbol{k})] \tag{19}$$

$$\int d^3x(\nabla\phi)^2 = \int d^3k\frac{\hbar\boldsymbol{k}^2}{2\omega}[a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) + a(\boldsymbol{k})a^\dagger(\boldsymbol{k}) + a(\boldsymbol{k})a(-\boldsymbol{k}) + a^\dagger(\boldsymbol{k})a^\dagger(-\boldsymbol{k})] \tag{20}$$

$$\int d^3xm^2\phi^2 = \int d^3k\frac{\hbar m^2}{2\omega}[a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) + a(\boldsymbol{k})a^\dagger(\boldsymbol{k}) + a(\boldsymbol{k})a(-\boldsymbol{k}) + a^\dagger(\boldsymbol{k})a^\dagger(-\boldsymbol{k})] \tag{21}$$

Adding them up we find

$$H = \frac{1}{2}\int d^3k\hbar\omega[a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) + a(\boldsymbol{k})a^\dagger(\boldsymbol{k})] = E_0 + \int d^3k\hbar\omega a^\dagger(\boldsymbol{k})a(\boldsymbol{k})$$

Similar manipulations lead to

$$\boldsymbol{P} = -\int d^3x\Pi\nabla\phi = \int d^3k\frac{\hbar\boldsymbol{k}}{2}[a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) + a(\boldsymbol{k})a^\dagger(\boldsymbol{k})] = \int d^3k\hbar\boldsymbol{k}a^\dagger(\boldsymbol{k})a(\boldsymbol{k})$$

In this case the constant piece $\delta(\boldsymbol{0})\int d^3k\hbar\boldsymbol{k}/2 = 0$ upon integrating over directions.

Note that in the future we will be assuming units where $\hbar = 1$. I left $\hbar \neq 1$ in this problem to show the familiar Planck condition $E = \hbar\omega = h\nu$ and the de Broglie relation $\boldsymbol{P} = \hbar\boldsymbol{k}$. A difference between Srednicki's and my conventions is that I normalize creation and annihilation operators so that $[a, a^\dagger] = \delta$, compared to his $[a_S, a_S^\dagger] = 2\omega(\boldsymbol{k})(2\pi)^3\delta$. (see S (3.29)) This explains the apparent difference between the above equations and S (3.19).