Quantum Field Theory

Solution Set 2

Due: Friday, 25 January 2019

Reading: Lecture Notes, Chapters 1,2; S, Sections 54-55.

5. Starting with the expansion of $\phi(x), \pi(x)$ in terms of creation and annihilation operators, show that the canonical commutation rules

$$\begin{aligned}[\phi(\boldsymbol{x},t),\phi(\boldsymbol{y},t)] &= [\pi(\boldsymbol{x},t),\pi(\boldsymbol{y},t)] = 0 \\ [\phi(\boldsymbol{x},t),\pi(\boldsymbol{y},t)] &= i\hbar\delta(\boldsymbol{x}-\boldsymbol{y}) \end{aligned} \tag{1}$$

follow from $[a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})] = \delta(\boldsymbol{p}-\boldsymbol{q})$ and $[a(\boldsymbol{p}), a(\boldsymbol{q})] = 0$. since the fields are at equal times.

Solution: The commutator of the integrands of two ϕ's involves

$$[(a_p e^{ix\cdot p} + a_p^\dagger e^{-ix\cdot p}), (a_p e^{iy\cdot q} + a_q^\dagger e^{-iy\cdot q})] = \delta(\boldsymbol{p}-\boldsymbol{q}) e^{i(\boldsymbol{x}-y)\cdot\boldsymbol{p}} - \delta(\boldsymbol{p}-\boldsymbol{q}) e^{-i(\boldsymbol{x}-y)\cdot\boldsymbol{p}} \tag{2}$$

Because the remaining integrand over $\boldsymbol{p}$ is even under $\boldsymbol{p} \to -\boldsymbol{p}$ the two twrms cancel. The same argument applies to two $\dot\phi = \pi$'s.If the first field in the commutator is a ϕ and the srcond is a $\dot\phi$, there is a relative minus between the a and $a^\dagger$ terms of the $\dot\phi$, so the two terms add and give $i\delta(\boldsymbol{x}-\boldsymbol{y})$ as desured. The factor of ω from the time derivative cancels the $1/\omega$ from the integrands of the fields, leaving a net factor $1/[2(2\pi)^3]$. The factors of 2π give the delta finction and the 12 is cancelled by the addition of two equal terms.

6. Recall that a Lorentz transformation of a scalar field is $\phi'(x) = \phi(\Lambda^{-1}x)$. In QFT, Lorentz transformations are represented by unitary operators $U(\Lambda)$ which accordingly transform the scalar field as

$$\phi'(x) \equiv U^{-1}(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x) \tag{3}$$

Show that this requirement implies that $a, a^\dagger$ transform as

$$U(\Lambda)^{-1}a(\boldsymbol{k})U(\Lambda) = \sqrt{\frac{\omega(\Lambda^{-1}\boldsymbol{k})}{\omega(\boldsymbol{k})}}\,a(\Lambda^{-1}\boldsymbol{k}), \qquad U(\Lambda)^{-1}a^\dagger(\boldsymbol{k})U(\Lambda) = \sqrt{\frac{\omega(\Lambda^{-1}\boldsymbol{k})}{\omega(\boldsymbol{k})}}\,a^\dagger(\Lambda^{-1}\boldsymbol{k})$$

[The notation $\Lambda^{-1}\mathbf{k}$ means the spatial components of the four vector $(\Lambda^{-1})^\mu{}_\nu k^\nu$, where $k^0 \equiv \sqrt{\mathbf{k}^2+m^2}$. Note that Srednicki chooses a different normalization for $a, a^\dagger$: $[a_S(\boldsymbol{q}), a_S^\dagger(\boldsymbol{p})] = (2\pi)^3 2\omega(\boldsymbol{p})\delta(\boldsymbol{q}-\boldsymbol{p})$.] Show that with his normalization the square root prefactor in the transformation law is absent.

Solution: It follows from the Lorentz invariance of the scalar product $x\cdot k = \boldsymbol{x}\cdot\boldsymbol{k} - t\omega(\boldsymbol{k})$ that $(\Lambda^{-1}x)\cdot k = x\cdot(\Lambda k)$. We want to change d^3k integration variables to d^3k', where $k^{i\prime} = \Lambda^i{}_j k^j + \Lambda^i{}_0\omega(\boldsymbol{k}) \equiv (\Lambda\boldsymbol{k})^i$. To do this consider

$$\int d^4k\,\delta(k\cdot k + m^2) = \int d^3k \frac{1}{d(k^2+m^2)/dk^0}\bigg|_{k^0=\omega(\boldsymbol{k})} = \int \frac{d^3k}{2\omega(\boldsymbol{k})} \tag{4}$$

where we integrated the delta function over k^0. The left side is manifestly Lorentz invariant so it follow that the right side is also Lorentz invariant. Therefore $d^3k/\omega(\boldsymbol{k}) = d^3k'`/\omega(\boldsymbol{k}'))$. With our normalization of the a's, the integrand of the expression for $\phi(\Lambda^{-1}x)$ involves

$$\frac{d^3k}{\sqrt{\omega(\boldsymbol{k})}} a(\boldsymbol{k}) e^{ik\cdot(\Lambda^{-1}x)} = \frac{d^3k'}{\sqrt{\omega(\boldsymbol{k}')}} \sqrt{\frac{\omega(\Lambda^{-1}\boldsymbol{k}')}{\omega(\boldsymbol{k}')}} a(\Lambda^{-1}\boldsymbol{k}') e^{ik'\cdot x}$$

Thus, comparing the a term of $U^{-1}\phi U$ to the right side with the prime removed from the dummy variable, we see

$$U(\Lambda)^{-1} a(\boldsymbol{k}) U(\Lambda) = \sqrt{\frac{\omega(\Lambda^{-1}\boldsymbol{k})}{\omega(\boldsymbol{k})}} a(\Lambda^{-1}\boldsymbol{k})$$

Since Srednicki's $a_S = (2\pi)^{3/2}\sqrt{2\omega(\boldsymbol{k})}a(\boldsymbol{k})$, it follows that

$$U(\Lambda)^{-1} a_S(\boldsymbol{k}) U(\Lambda) = a_S(\Lambda^{-1}\boldsymbol{k})$$

That $U(\Lambda)^{-1}a_S^\dagger(\boldsymbol{k})U(\Lambda) = a_S^\dagger(\Lambda^{-1}\boldsymbol{k})$ follows immediately by hermitian conjugating both sides and using $U^\dagger = U^{-1}$. Using this formula but with $\Lambda \to \Lambda^{-1}$ shows that $U(\Lambda)|k_1 \dots k_n\rangle_S = |\Lambda k_1 \dots \Lambda k_n\rangle_S$

7. **The Free Complex Scalar Field**. A non-hermitian ("complex") scalar field has the expansion

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p})}} \left(a(\mathbf{p})e^{ip\cdot x} + b^\dagger(\mathbf{p})e^{-ip\cdot x}\right), \qquad \phi^\dagger \neq \phi \tag{5}$$

Here $p \cdot x \equiv \boldsymbol{p}\cdot\boldsymbol{x} - \omega(\boldsymbol{p})t$; $a^\dagger(\mathbf{p})$ creates a spin 0 particle; and $b^\dagger(\mathbf{p})$ creates the associated antiparticle. Their commutation relations are

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = [b(\mathbf{p}), b^\dagger(\mathbf{p}')] = \delta(\mathbf{p}' - \mathbf{p}),$$

with all other commutators vanishing.

(a) Show from (5) that ϕ and $\dot{\phi}^\dagger$ satisfy the equal time commutation relations

$$[\phi(\mathbf{x}, t), \dot{\phi}^\dagger(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}),$$

and that ϕ satisfies the Klein-Gordon Equation

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2} - m^2\right)\phi(x) = 0.$$

Solution:

$$
\begin{aligned}
[\phi(\boldsymbol{x},t),\dot\phi^\dagger(\boldsymbol{y},t)] &= \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega}}\int \frac{d^3p'(-i\omega')}{(2\pi)^{3/2}\sqrt{2\omega'}} \\
&\qquad [a(\boldsymbol{p})e^{ix\cdot p}+b^\dagger(\boldsymbol{p})e^{-ix\cdot p}, b(\boldsymbol{p}')e^{iy\cdot p'}-a^\dagger(\boldsymbol{p}')e^{-iy\cdot p'}] \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega}\sqrt{2\omega'}} d^3p'(-i\omega')\delta(\boldsymbol{p}-\boldsymbol{p}')\left(-e^{i(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{p}}-e^{-i(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{p}}\right) \\
&= i\int \frac{d^3p}{2(2\pi)^3}\left(e^{i(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{p}}+e^{-i(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{p}}\right) = i\delta(\boldsymbol{x}-\boldsymbol{y}) \qquad (6)
\end{aligned}
$$

$$
(\nabla^2-\frac{\partial^2}{\partial t^2}-m^2)\phi = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(\boldsymbol{p})}}(-m^2-p^2)\left(a(\boldsymbol{p})e^{ip\cdot x}+b^\dagger(\boldsymbol{p})e^{-ip\cdot x}\right)=0
$$

because $p^0=\sqrt{m^2+\boldsymbol{p}^2}$.

(b) Alternatively we can work with hermitian fields ϕ_1,ϕ_2 defined by $\phi=(\phi_1+i\phi_2)/\sqrt{2}$. Show that

$$[\phi_k(\mathbf{x},t),\dot\phi_l(\mathbf{y},t)] = i\delta_{kl}\delta^3(\mathbf{x}-\mathbf{y}).$$

Thus $\dot\phi_k\equiv\pi_k$ is the conjugate momentum to ϕ_k, and the Hamiltonian is just the sum of two commuting terms, one for each of the hermitian fields ϕ_1,ϕ_2:

$$H=\int d^3x\frac{1}{2}\sum_k(\pi_k^2+(\nabla\phi_k)^2+m^2\phi_k^2).$$

Solution: Solve $\phi_1=(\phi+\phi^\dagger)/\sqrt{2}$ and $\phi_2=-i(\phi-\phi^\dagger)/\sqrt{2}$. Then

$$[\phi_1,\dot\phi_2]=i[\phi,\dot\phi^\dagger]/2-i[\phi^\dagger,\dot\phi]/2=-\delta/2+\delta/2=0.$$

Similarly, $[\phi_2,\dot\phi_1]=0$.

$$[\phi_1,\dot\phi_1]=[\phi,\dot\phi^\dagger]/2+[\phi^\dagger,\dot\phi]/2=i\delta/2+i\delta/2=i\delta$$

and similarly for $[\phi_2,\dot\phi_2]$.

(c) Now returning to the original non-hermitian field ϕ, and defining $\pi=(\dot\phi_1^\dagger-i\dot\phi_2^\dagger)/\sqrt{2}$, show that

$$H=\int d^3x(\pi\pi^\dagger+\nabla\phi^\dagger\nabla\phi+m^2\phi^\dagger\phi),$$

and the equal time commutation relations become

$$[\phi(\boldsymbol{x},t),\pi(\boldsymbol{y},t)]=i\delta^3(\mathbf{x}-\mathbf{y}). \qquad (7)$$

Solution: First note:

$$\pi\pi^\dagger = \frac{1}{2}(\dot\phi_1^2 + \dot\phi_2^2) = \frac{1}{2}\sum_k \pi_k^2 \tag{8}$$

$$\phi^\dagger\phi = \frac{1}{2}\sum_k \phi_k^2 \tag{9}$$

$$\nabla\phi^\dagger\cdot\nabla\phi = \frac{1}{2}\sum_k \nabla\phi_k\cdot\nabla\phi_k \tag{10}$$

Then comparing with the Cartesian form for H we see

$$H = \int d^3x(\pi\pi^\dagger + \nabla\phi^\dagger\nabla\phi + m^2\phi^\dagger\phi)$$

and the equal time commutation relations become

$$[\phi(\boldsymbol{x},t),\pi(\boldsymbol{y},t)] = \frac{1}{2}\sum_k([\phi_k(\boldsymbol{x},t),\dot\phi_k(\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{11}$$

8. **Coupling Classical Electromagnetism to a Quantum Scalar Field**

(a) The minimal substitution rule for coupling an external electromagnetic field in a gauge invariant way is the substitution $\partial \to \partial - iqA$ in the Klein-Gordon equation. [Note that the appearance of i in this rule is why electromagnetism must couple to a complex (*i.e.* non-hermitian field).] Show that the resulting equation follows from the Heisenberg equations derived, using (7) and the Hamiltonian

$$H = \int d^3x(\pi\pi^\dagger + (\nabla + iq\mathbf{A})\phi^\dagger(\nabla - iq\mathbf{A})\phi + m^2\phi^\dagger\phi + iqA_0(\pi\phi - \phi^\dagger\pi^\dagger)),$$

Solution: Writing out the Heisenberg equations $i\dot A = [A,H]$ for $\phi, \pi^\dagger$:

$$\begin{aligned}
i\dot\phi = [\phi,H] &= i(\pi^\dagger + iqA_0)\phi \\
&\quad\to \quad \pi^\dagger = \dot\phi - iqA_0\phi && (12)\\
i\dot\pi^\dagger = [\pi^\dagger,H] &= i(\nabla - iq\mathbf{A})^2\phi - im^2\phi + i^2qA_0\pi^\dagger) \\
&\quad\to \quad \dot\pi^\dagger - iqA_0\pi^\dagger = (\nabla - iq\mathbf{A})^2\phi - m^2\phi && (13)\\
(\partial_0 - iqA_0)^2\phi &= (\nabla - iq\mathbf{A})^2\phi - m^2\phi \\
&\quad\to \quad (\partial - iqA)^2\phi - m^2\phi = 0 && (14)
\end{aligned}$$

as desired.

(b) We shall soon learn that gauge invariant time evolution implies that a conserved current can be defined in terms of the change in the Schrödinger picture Hamiltonian, under a small change in the potentials with canonical variables fixed (i.e. ϕ, its *spatial* derivatives and π are held fixed):

$$U^\dagger(t)\delta H_S U(t) = -\int d^3x j_\mu(\boldsymbol{x},t)\delta A^\mu(\boldsymbol{x},t).$$

Here $U(t)$ converts Schrödinger to Heisenberg picture. From the Hamiltonian in part (a) use this principle and the Heisenberg equations to obtain the expression for the current

$$j_\mu(x) = -iq(\phi^\dagger\partial_\mu\phi - (\partial_\mu\phi^\dagger)\phi) - 2q^2 A_\mu\phi^\dagger\phi.$$

Confirm that $\partial_\mu j^\mu = 0$ as a consequence of the Klein-Gordon equation coupled to A_μ.

Solution:

$$\delta H = \int d^3x iq\delta A^i[\phi^\dagger\nabla_i\phi - (\nabla_i\phi^\dagger)\phi - 2iqA_i\phi^\dagger\phi] - \int d^3x iq\delta A^0(\pi\phi - \pi^\dagger\phi^\dagger) \qquad (15)$$

This implies

$$\begin{aligned} j_i &= -iq(\phi^\dagger\partial_i\phi - (\partial_i\phi^\dagger)\phi) - 2q^2 A_i\phi^\dagger\phi & (16) \\ j_0 &= iq\delta(\pi\phi - \pi^\dagger\phi^\dagger) \\ &= iq(\dot\phi^\dagger + iqA_0\phi^\dagger)\phi - iq(\dot\phi - iqA_0\phi)\phi^\dagger = -iq(\phi^\dagger\dot\phi - \dot\phi^\dagger\phi) - 2q^2A_0\phi^\dagger\phi) & (17) \end{aligned}$$

which are just the space and time components of

$$j_\mu = -iq(\phi^\dagger\partial_\mu\phi - (\partial_\mu\phi^\dagger)\phi) - 2q^2A_\mu\phi^\dagger\phi.$$

to confirm that $\partial_\mu j^\mu = 0$ we simply calculate

$$\partial_\mu j^\mu = -iq(\phi^\dagger\partial^2\phi - (\partial^2\phi^\dagger)\phi) - 2q^2\partial_\mu(A^\mu\phi^\dagger\phi).$$

From KG eq,

$$\partial^2\phi = 2iqA^\mu\partial_\mu\phi + iq\phi\partial\cdot A + (m^2 + q^2A^2)\phi.$$

Then

$$(\phi^\dagger\partial^2\phi - (\partial^2\phi^\dagger)\phi) = 2iqA^\mu\partial_\mu(\phi^\dagger\phi) + 2iq\phi^\dagger\phi\partial\cdot A = 2iq\partial_\mu(A^\mu\phi^\dagger\phi).$$

Putting this in the expression for $\partial_\mu j^\mu$ shows that the latter is zero..

(c) Work out the charge $Q = \int d^3x j^0$ in terms of creation and annihilation operators for the case of zero external field ($A_\mu = 0$).

Solution:

$$
\begin{aligned}
Q &= \int d^3x j^0 = iq\int d^3x \frac{d^3pd^3p'}{(2\pi)^3\sqrt{4\omega\omega'}}\Big[(a^\dagger(\boldsymbol{p})e^{-ix\cdot p}+b(\boldsymbol{p})e^{ix\cdot p})(-i\omega')(a(\boldsymbol{p'})e^{ix\cdot p'}-b^\dagger(\boldsymbol{p'})e^{-ix\cdot p'}) \\
&\quad -(i\omega)(a^\dagger(\boldsymbol{p})e^{-ix\cdot p}-b(\boldsymbol{p})e^{ix\cdot p})(a(\boldsymbol{p'})e^{ix\cdot p'}+b^\dagger(\boldsymbol{p'})e^{-ix\cdot p'})\Big] \qquad (18) \\
&= iq\int\frac{d^3p}{2\omega}\Big[(a^\dagger(\boldsymbol{p})a(\boldsymbol{p})-b(\boldsymbol{p})b^\dagger(\boldsymbol{p})-a^\dagger(\boldsymbol{p})b^\dagger(-\boldsymbol{p})e^{2i\omega t}+b(\boldsymbol{p})a^\dagger(-\boldsymbol{p})e^{-2i\omega t})(-i\omega) \\
&\quad -(i\omega)(a^\dagger(\boldsymbol{p})a(\boldsymbol{p}-b(\boldsymbol{p})b^\dagger(\boldsymbol{p})+a^\dagger(\boldsymbol{p})b^\dagger(-\boldsymbol{p})e^{2i\omega t}-b(\boldsymbol{p})a^\dagger(-\boldsymbol{p})e^{-2i\omega t})\Big] \qquad (19) \\
&= q\int d^3p[a^\dagger(\boldsymbol{p})a(\boldsymbol{p})-b(\boldsymbol{p})b^\dagger(\boldsymbol{p})] = q\int d^3p[a^\dagger(\boldsymbol{p})a(\boldsymbol{p})-b^\dagger(\boldsymbol{p})b(\boldsymbol{p})] - \text{an infinite constant}
\end{aligned}
$$

showing that $a^\dagger$ and $b^\dagger$ create particles of charge q and $-q$ respectively.