Quantum Field Theory

Solution Set 3

Due: 1 February 2019

Reading: Lecture Notes, Chapter 1; S, Sections 54-55.

9. Show that in Coulomb gauge, the canonical free EM Hamiltonian reduces to

$$H_{Coulomb} = \int d^3x \frac{1}{2} \sum_{l=1}^{3} \left[\Pi^l \Pi^l + \nabla A^l \cdot \nabla A^l \right] \tag{1}$$

where the l sum is just over the components of the vector operators $\boldsymbol{\Pi}, \boldsymbol{A}$. Then show, by substituting the expansions for Π^l, A^l in terms of creation and annihilation operators into this formula, that the Hamiltonian is just $H_{Coul} = \int d^3k |\boldsymbol{k}| \sum_l a_l^\dagger(\boldsymbol{k}) a_l(\boldsymbol{k})$ and also that the total momentum is $\boldsymbol{P} = \int d^3k \boldsymbol{k} \sum_l a_l^\dagger(\boldsymbol{k}) a_l(\boldsymbol{k})$.

Solution: We use the Coulomb gauge condition $\nabla \cdot \boldsymbol{A} = 0$ to write

$$\boldsymbol{B}^2 \;=\; (\nabla \times \boldsymbol{A}) \cdot (\nabla \times \boldsymbol{A}) = [(\nabla \times \boldsymbol{A}) \times \nabla] \cdot \boldsymbol{A}) = \nabla_k \boldsymbol{A} \cdot \nabla_k \boldsymbol{A} - \nabla_k A_i \nabla_i A_k \tag{2}$$

The spatial integral of the second term can ve integrated by parts and becomes $(\nabla \cdot A)^2 = 0$. The first term just gives the desired second term of the Hamiltonian. Then $\boldsymbol{\Pi} = -\boldsymbol{E} = \dot{\boldsymbol{A}} + \nabla A^0 = \dot{\boldsymbol{A}}$ since the Coulomb gauge condition and Gauss law implies $\nabla^2 A^0 = 0$, which for bounded A^0 implies that $\nabla A^0 = 0$. This establishes the simplification of H. Each component of $\boldsymbol{A}$ now enters the Hamiltonian precisely like a massless scalar field. Similarly the momentum density can be written

$$\boldsymbol{E} \times (\nabla \times \boldsymbol{A}) = -\dot{\boldsymbol{A}}_k \nabla A_k - (\boldsymbol{E} \cdot \nabla) \boldsymbol{A} \tag{3}$$

and the integral of the second term can be integrated by parts turning it into $\nabla \cdot \boldsymbol{E} \boldsymbol{A} = 0$ by Gauss law. Thus we can immediately write, without further calculation

$$H = \int d^3k |\boldsymbol{k}| \boldsymbol{a}^\dagger(\boldsymbol{k}) \cdot \boldsymbol{a}(\boldsymbol{k}), \qquad \boldsymbol{P} = \int d^3k \boldsymbol{k} \boldsymbol{a}^\dagger(\boldsymbol{k}) \cdot \boldsymbol{a}(\boldsymbol{k}) \tag{4}$$

10. **Canonical Energy Momentum Tensor**. Given a general Lagrangian density $\mathcal{L}(\partial_\mu \phi_k, \phi_k)$, the canonical energy momentum tensor is defined by

$$T^{\mu\nu}_{\text{Can}} \;=\; -\sum_k \partial^\mu \phi_k \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_k)} + \eta^{\mu\nu} \mathcal{L} \tag{5}$$

a) Use the field equations implied by Hamilton's principle, applied to $S = \int d^4x \mathcal{L}$, to prove that this construction is automatically conserved: $\partial_\nu T^{\mu\nu} = 0$.

Solution: Hamilton's principle applied to S gives

$$0 \;=\; \int d^4x \sum_k \delta\phi_k \left[\frac{\partial\mathcal{L}}{\partial\phi_k} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)}\right] \tag{6}$$

for arbitrary $\delta\phi_k$, implying that the quantity in square brackets vanishes for each k. On the other hand the conservation law on $T^{\mu\nu}_{\rm Can}$ reads

$$\begin{aligned} 0 \;&=\; -\sum_k \partial_\nu\partial^\mu\phi_k \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_k)} - \sum_k \partial^\mu\phi_k\partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_k)} + \left[\partial^\mu\phi_k \frac{\partial\mathcal{L}}{\partial\phi_k} + \partial^\mu\partial_\nu\phi_k \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_k)}\right] \\ &=\; -\sum_k \partial^\mu\phi_k \left[\partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_k)} - \frac{\partial\mathcal{L}}{\partial\phi_k}\right] = 0 \end{aligned} \tag{7}$$

by the equations implied by Hamilton's principle.

b) While the canonical construction is automatically conserved, it is *not* automatically symmetric. Construct the canonical energy momentum tensor for the free electromagnetic field and show that it is not symmetric, i.e. $T^{\mu\nu}_{\rm Can} \neq T^{\nu\mu}_{\rm Can}$

Solution: For the free em field $\mathcal{L} = -F_{\mu\nu}F^{\mu\nu}/4$. We need

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} \tag{8}$$

The 4 cancels because there are 2 F's and 2 terms in each F. Then plugging in we find

$$T^{\mu\nu}_{\rm Can} = \partial^\mu A_\rho F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{9}$$

The first term is clearly not symmetric under $\mu \leftrightarrow \nu$ and the second term is symmetric, so $T^{\mu\nu}_{\rm Can} \neq T^{\nu\mu}_{\rm Can}$.

c) In class we used a definition of the EM energy momentum tensor which is symmetric. Show that it differs from the canonical one by a term that does not contribute to the total energy and momentum (when the field equations are satisfied). In other words, show that $T^{\mu 0}_{\rm Can} - T^{\mu 0}_{\rm Class}$ is a total *spatial* derivative. The symmetric energy momentum tensor is sometimes called the "improved" energy momentum tensor. It is the improved one that must be used in the construction of the Lorentz generator density $\mathcal{M}^{\mu\nu\rho}$.

Solution: Consulting lecture notes we recall

$$T^{\mu\nu}_{\rm Class} \;=\; F^\mu{}_\rho F^{\nu\rho} - \eta^{\mu\nu}\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} \tag{10}$$

so the difference is

$$\begin{aligned} T^{\mu\nu}_{\rm Can} - T^{\mu\nu}_{\rm Class} \;&=\; (\partial^\mu A_\rho - F^\mu{}_\rho)F^{\nu\rho} = -(\partial_\rho A^\mu)F^{\nu\rho} = -\partial_\rho(A^\mu F^{\nu\rho}) + A^\mu\partial_\rho F^{\nu\rho} \\ &=\; -\partial_\rho(A^\mu F^{\nu\rho}) \end{aligned} \tag{11}$$

by Maxwell's equations. This is a total space-time derivative but not necessarily a spatial derivative. However for $\nu = 0$, the antisymmetry of $F^{\nu\rho}$ means $F^{0\rho} \neq 0$ only for spatial components. This means that the alternative tensors give the same total energy and momentum operators. As mentioned above the angular momentum and boost generators require the improved symmetric energy momentum tensor for their construction.

11. In class we obtained the angular momentum of the EM field

$$\boldsymbol{J} \;=\; \int d^3x \sum_k E_k(\boldsymbol{x} \times \boldsymbol{\nabla}) A_k + \int d^3k a_b^\dagger(\boldsymbol{k}) a_a(\boldsymbol{k}) \boldsymbol{S}_{ab} \tag{12}$$

where the 2×2 photon "spin" matrix is given by $\boldsymbol{S}_{ab} = i\boldsymbol{\epsilon}_a \times \boldsymbol{\epsilon}_b^*$.

a) Evaluate the action of the first "orbital" term on a single photon state $a_c^\dagger(\boldsymbol{k})|0\rangle$ and show that it doesn't contribute to the photon helicity.

Solution: Applying $\boldsymbol{J}$ to a one photon state we can write

$$\boldsymbol{J} a_a^\dagger(\boldsymbol{k})|0\rangle = [\boldsymbol{J}, a_a^\dagger(\boldsymbol{k})]|0\rangle + a_a^\dagger(\boldsymbol{k}) \boldsymbol{J}|0\rangle \tag{13}$$

The second term should vanish because the vacuum is rotationally invariant. The action on the one photon state is therefore given by the commutator terms, Call the orbital term $\boldsymbol{L}$. We want to calculate $[\boldsymbol{L}, a_a^\dagger(\boldsymbol{k}]$ for which we need

$$\begin{aligned}
[\boldsymbol{A}(\boldsymbol{x}), a_a^\dagger(\boldsymbol{k})] &= \frac{\boldsymbol{\epsilon}_a(\boldsymbol{k})}{(2\pi)^{3/2}\sqrt{2|\boldsymbol{k}|}} e^{ik\cdot x} \\
[\boldsymbol{E}(\boldsymbol{x}), a_a^\dagger(\boldsymbol{k})] &= \frac{i|\boldsymbol{k}|\boldsymbol{\epsilon}_a(\boldsymbol{k})}{(2\pi)^{3/2}\sqrt{2|\boldsymbol{k}|}} e^{ik\cdot x} \\
[\boldsymbol{L}, a_a^\dagger(\boldsymbol{k})] &= \int d^3x \frac{\epsilon_a^k(\boldsymbol{k})}{(2\pi)^{3/2}\sqrt{2|\boldsymbol{k}|}} \left(E_k(\boldsymbol{x} \times \nabla) e^{ik\cdot x} + i|\boldsymbol{k}| e^{ik\cdot x} (\boldsymbol{x} \times \nabla) A_k\right) \\
&= \int d^3x (\boldsymbol{x} \times i\boldsymbol{k}) \frac{\epsilon_a^k(\boldsymbol{k})}{(2\pi)^{3/2}\sqrt{2|\boldsymbol{k}|}} \left(E_k e^{ik\cdot x} - i|\boldsymbol{k}| e^{ik\cdot x} A_k\right)
\end{aligned} \tag{14}$$

where we have integrated the second term by parts. The last expression is orthogonal to $\boldsymbol{k}$ and hence will not contribute to the helicity.

b) For $\boldsymbol{k}$ parallel to the z-axis $\boldsymbol{k} = k\hat{z}$ and the choices $\epsilon_1 = (1, i, 0)/\sqrt{2}$ and $\epsilon_2 = (1, -i, 0)/\sqrt{2}$, calculate the three matrices S^x, S^y, S^z.

Solution: We evaluate in turn

$$
\begin{aligned}
\boldsymbol{S}_{11} &= \frac{i}{2}(\hat{x} + i\hat{y}) \times (\hat{x} - i\hat{y}) = \frac{1}{2}(\hat{x} \times \hat{y} - \hat{y} \times \hat{x}) = \hat{z} \\
\boldsymbol{S}_{22} &= \frac{i}{2}(\hat{x} - i\hat{y}) \times (\hat{x} + i\hat{y}) = \frac{1}{2}(-\hat{x} \times \hat{y} + \hat{y} \times \hat{x}) = -\hat{z} \\
\boldsymbol{S}_{12} &= \frac{i}{2}(\hat{x} + i\hat{y}) \times (\hat{x} + i\hat{y}) = \frac{1}{2}(-\hat{x} \times \hat{y} - \hat{y} \times \hat{x}) = 0 \\
\boldsymbol{S}_{21} &= \frac{i}{2}(\hat{x} - i\hat{y}) \times (\hat{x} - i\hat{y}) = \frac{1}{2}(\hat{x} \times \hat{y} + \hat{y} \times \hat{x}) = 0
\end{aligned}
$$

Or in matrix notation $\boldsymbol{S} = \hat{z}\sigma_3$, where σ_3 is the diagonal Pauli matrix. Under a rotation that points the photon momentum in the direction $\hat{k} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ the spin matrix goes to $\hat{k}\sigma_3$. This peculiar feature of a particle's spin is characteristic of massless particles: the helicity of a massless particle is a Lorentz invariant!