Quantum Field Theory

Problem Set 4

Due: 8 February 2019

Reading: S, 33-40; Lecture Notes, Chapter 2,3.

12. **Representations for γ^μ.** For this exercise we consider representations of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = -2\eta_{\mu\nu}$ for general spacetime dimension D: so there are D anticommuting matrices γ^μ, $\mu = 0, 1, \ldots, D-1$. Remember that γ^0 must be hermitian and the spatial components γ^k must be antihermitian.

a) For D even, show that the matrix $\gamma_{D+1} \equiv i\gamma^0\gamma^1 \cdots \gamma^{D-1}$ anticommutes with all of the γ^μ. This means that given a representation of γ^μ for D even, one can treat γ_{D+1} or $i\gamma_{D+!}$ as an extra spatial γ. Show that it is hermitian for $D = 4$ but antihermitian for $D = 2, 6$.

Solution: When D is even, γ_{D+1} is the product of each of the D gamma matrices. Pick a γ^μ. then γ_{D+1} contains $D-1$ γ^ν with $\nu \neq \mu$ and γ^μ anticommutes with each of these, and since there are an odd number, anticommutes with the product of all of them. Of course γ^μ commutes with itself, so all together anticommutes with γ_{D+1}. For $D = 4$.

$$(i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger = -i(-)\gamma^3\gamma^2\gamma^1\gamma^0 = -i\gamma^2\gamma^1\gamma^0\gamma^3 = -i\gamma^1\gamma^0\gamma^2\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{1}$$

But for $D = 6$

$$(i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5)^\dagger = -i(-)\gamma^5\gamma^4\gamma^3\gamma^2\gamma^1\gamma^0 = -i\gamma^3\gamma^2\gamma^1\gamma^0\gamma^4\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5 \tag{2}$$

b) Show that in D space-time dimensions the γ matrices must be at least of size $2^{D/2} \times 2^{D/2}$ for D even and at least $2^{(D-1)/2} \times 2^{(D-1)/2}$ for D odd.

Solution: When D is even we can find $D/2$ commuting matrices by pairing them up: $\gamma^0\gamma^1, \gamma^2\gamma^3, ..., \gamma^{D-2}\gamma^{D-1}$. Since each of these commuting matrices has at least two distinct eigenvalues ± 1 the matrices have to be at least $2^{D/2}$ dimensional. If D is odd, $D-1$ is even so we can find $D-1$ gamma matrices of dimension $2^{(D-1)/2}$. But then the product of all $D-1$ of these gamma matrices always gives one more so $2^{(D-1)/2}$ is the minimal matrix size in this case.

c) In addition to the standard and chiral representations, it is sometimes useful to have a Majorana representation, for which all gamma matrices are imaginary $\gamma^{\mu *} = -\gamma^\mu$. Find such a representation for $D = 4$, by selecting 4 of the 5 anticommuting matrices γ^μ, γ_5 in the standard representation and, if necessary, multiplying some of them by i.

Solution: There are of course many possibilities. One way is to simply rearrange the standard rep γ's we already have. For example $\gamma^0_{\rm Maj} = i\gamma^1$, $\gamma^1_{\rm Maj} = i\gamma^0$, $\gamma^2_{\rm Maj} =$

$\gamma^2, \gamma^3_{\text{Maj}} = i\gamma_5$ does the trick. Note that we still have the hermiticity property that $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.

13. **Lorentz Covariance of Dirac Equation** Let $\Lambda^\mu{}_\nu$ represent a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. In the new frame, the Dirac equation is

$$\frac{1}{i}\gamma^\mu \partial'_\mu \psi'(x') + m\psi'(x') = 0$$

Note that the *same* gamma matrices are used in every Lorentz frame! Our goal is to express $\psi'(x')$ in terms of $\psi(x)$.

(a) Let $\sigma^{mn} = \frac{1}{2}i[\gamma^m, \gamma^n]$, where $m, n = 1, 2, 3$ are spatial indices. Using the standard or chiral representation, show that $\sigma^{mn} = \epsilon_{mnk}\Sigma^k$ where $\mathbf{\Sigma}/2$ is the spin matrix for the Dirac equation.

Solution:

$$[\gamma^m, \gamma^n] = -\begin{pmatrix} [\sigma^m, \sigma^n] & 0 \\ 0 & [\sigma^m, \sigma^n] \end{pmatrix} = -2i\epsilon_{mnk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv -2i\epsilon_{mnk}\Sigma^k$$

(b) This result suggests that $M^{\mu\nu} = \frac{1}{4}i[\gamma^\mu, \gamma^\nu] \equiv \frac{1}{2}\sigma^{\mu\nu}$ is the generator of Lorentz transformations. Show that $M^{\mu\nu}$ satisfies the Lorentz algebra commutation relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\rho\nu} - \eta^{\nu\sigma}M^{\rho\mu}), \qquad (3)$$

It will be helpful to first show that the commutators of M with γ are those with γ a four-vector, *i.e.*

$$[M^{\mu\nu}, \gamma^\rho] = i(\eta^{\mu\rho}\gamma^\nu - \eta^{\nu\rho}\gamma^\mu).$$

Solution: Use the Clifford algebra on

$$[[\gamma^\mu, \gamma^\nu], \gamma^\rho] = \gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\nu\gamma^\mu = -4\gamma^\mu\eta^{\nu\rho} + 4\gamma^\nu\eta^{\mu\rho}$$

to move the γ^ρ's in the third and fourth terms all the way to the right. This establishes

$$[M^{\mu\nu}, \gamma^\rho] = i(\gamma^\nu\eta^{\mu\rho} - \gamma^\mu\eta^{\nu\rho})$$

. Then $[M^{\mu\nu}, \gamma^\rho\gamma^\sigma]$ can be easily obtained verifying the Lorentz algebra.

(c) Exploit these results to show that

$$\psi'(x') = e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\psi(\Lambda^{-1}x')$$

is a solution of the Dirac equation in the new frame provided $\psi(x)$ is a solution in the original frame and the matrix λ is related to Λ by

$$\Lambda^\mu{}_\nu = (e^{-\lambda})^\mu{}_\nu.$$

[Note that because of the group property of the $\sigma^{\mu\nu}$ established in the second part of this problem, it is sufficient to demonstrate this for infinitesimal λ.]

Solution:

$$\partial'_\mu \gamma^\mu e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\psi(\Lambda^{-1}x') = e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}(\Lambda^{-1})^\alpha{}_\mu e^{i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\gamma^\mu e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\partial_\alpha\psi'$$

. The result follows if

$$e^{i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\gamma^\mu e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4} = (\Lambda^{-1})_\beta{}^\mu\gamma^\beta$$

. We can easily establish this last equality for infinitesimal transformations:

$$(\Lambda^{-1})^{\beta\mu} = \eta^{\beta\mu} - G^{\beta\mu}$$

:

$$e^{i\lambda_{\rho\tau}\sigma^{\rho\tau}/4}\gamma^\mu e^{-i\lambda_{\rho\tau}\sigma^{\rho\tau}/4} = \gamma^\mu - \frac{i}{4}[\gamma^\mu, \lambda_{\rho\tau}\sigma^{\rho\tau}] = \gamma^\mu + \frac{1}{2}(\gamma^\rho\lambda_\rho{}^\mu - \gamma^\rho\lambda^\mu{}_\sigma) = \gamma^\mu + \gamma^\rho\lambda_\rho{}^\mu$$

where we used antisymmetry of λ. We get the desired result if $\lambda^{\beta\mu} = -G^{\beta\mu}$ which is the infinitesimal version of $\Lambda = e^{-\lambda}$. Note that $G^{\mu\nu}$ is antisymmetric in its indices.

14. **Helicity basis for spin 1/2 particles.** Helicity is defined as the component of angular momentum of a particle along its momentum, *i.e.* $h = \boldsymbol{p}\cdot\boldsymbol{J}/|\boldsymbol{p}|$ with $\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{p} + \boldsymbol{\sigma}/2$. Note that since $\boldsymbol{p}\cdot(\boldsymbol{r}\times\boldsymbol{p}) = 0$, we may write, more simply, $h = \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}/2$, where $\hat{\boldsymbol{p}}$ is a unit vector parallel to $\boldsymbol{p}$.

(a) Show that the eigenvalues of h are $\pm 1/2$.

Solution: $h = \boldsymbol{p}\cdot\boldsymbol{\sigma}/2|\boldsymbol{p}|$ satisfies $h^2 = 1/4$ so its eigenvalues are $\pm 1/2$.

(b) Consider a single particle state $|\boldsymbol{p}, h\rangle$ with momentum $\boldsymbol{p}$ and helicity h. Show that rotations leave the helicity of the state unchanged.

Solution: Under rotations, generated by $\boldsymbol{J}$, both $\boldsymbol{p}$ and $\boldsymbol{\sigma}$ transform as vectors. Thus h, the scalar product of two vectors, commutes with rotations, implying that rotations do not change the helicity.

(c) For $\boldsymbol{p} = p\hat{\mathbf{z}}$, $h = \sigma_z/2$ and we may take $|p\hat{\mathbf{z}}, h = \pm 1/2\rangle \equiv |p\hat{\mathbf{z}}, \sigma_z = \pm 1\rangle$ and we fix the ambiguity at $\boldsymbol{p} = 0$ by defining $|\mathbf{0}, h = \pm 1/2\rangle \equiv |\mathbf{0}, \sigma_z = \pm 1\rangle$. According to (b), we may define

$$|\boldsymbol{p}, h = \pm 1/2\rangle \equiv R_0(\boldsymbol{p})|p\hat{\mathbf{z}}, \sigma_z = \pm 1\rangle$$

where $R_0(\boldsymbol{p})$ is a standardized rotation that takes $p\hat{\mathbf{z}}$ into $\boldsymbol{p}$. Let θ, ϕ be the polar angles of $\boldsymbol{p}$. Then take

$$R_0(\boldsymbol{p}) \equiv e^{-i\phi J_z}e^{-i\theta J_y}e^{+i\phi J_z}.$$

With this definition and using spinor notation

$$|\boldsymbol{p}, \sigma_z = 1\rangle \equiv |\boldsymbol{p}\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\boldsymbol{p}, \sigma_z = -1\rangle \equiv |\boldsymbol{p}\rangle \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

show that

$$|\boldsymbol{p}, h = 1/2\rangle = |\boldsymbol{p}\rangle \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \equiv |\boldsymbol{p}\rangle \chi_{1/2}(\boldsymbol{p}) \tag{4}$$

$$|\boldsymbol{p}, h = -1/2\rangle = |\boldsymbol{p}\rangle \begin{pmatrix} -e^{-i\phi}\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} \equiv |\boldsymbol{p}\rangle \chi_{-1/2}(\boldsymbol{p}). \tag{5}$$

Solution: We evaluate

$$R_0(\boldsymbol{p})|p\hat{z}\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\boldsymbol{p}\rangle \chi_{1/2}(\boldsymbol{p})$$

where

$$\begin{aligned} \chi^{1/2}(\boldsymbol{p}) &= e^{-i\phi\sigma^3/2} e^{-i\theta\sigma^2/2} e^{i\phi\sigma^z/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= e^{i\phi/2} e^{-i\phi\sigma^3/2} (\cos\theta/2 - i\sigma^2 \sin\theta/2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \end{aligned} \tag{6}$$

Similarly we find

$$\begin{aligned} \chi_{-1/2}(\boldsymbol{p}) &= e^{-i\phi\sigma^3/2} e^{-i\theta\sigma^2/2} e^{i\phi\sigma^z/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= e^{-i\phi/2} e^{-i\phi\sigma^3/2} (\cos\theta/2 - i\sigma^2 \sin\theta/2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -e^{-i\phi}\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} \end{aligned} \tag{7}$$

(d) Using the results of (c) prove the following identities:

$$\chi_\lambda(\boldsymbol{p}) = -ie^{i\lambda\pi + 2i\lambda\phi} \chi_{-\lambda}(-\boldsymbol{p}) \tag{8}$$

$$i\sigma^2 \chi_\lambda(\boldsymbol{p}) = e^{2i\lambda\phi} \chi^*_\lambda(-\boldsymbol{p}) \tag{9}$$

Solution: Putting $\theta_{-\boldsymbol{p}} = \pi - \theta_{\boldsymbol{p}}$ and $\phi_{-\boldsymbol{p}} = \pi + \phi_{\boldsymbol{p}}$, we have $\chi_{1/2}(-\boldsymbol{p}) = -e^{i\phi}\chi_{-1/2}(\boldsymbol{p})$ and $\chi_{-1/2}(-\boldsymbol{p}) = e^{-i\phi}\chi_{1/2}(\boldsymbol{p})$. We easily check that these two relations imply $\chi_\lambda(\boldsymbol{p}) = -ie^{i\pi\lambda + 2i\lambda\phi}\chi_{-\lambda}(-\boldsymbol{p})$. First evaluate $i\sigma_2\chi_\lambda(\boldsymbol{p}) = (-)^{\lambda+1/2}\chi^*_{-\lambda}(\boldsymbol{p})$. Then use the first identity to get the second identity.

15. Our discussion of the anticommutators of the creation and annihilation operators used in second quantization in class was a bit heuristic. Show that the concrete definition of $b_\alpha^\dagger$ acting on the occupation number basis,

$$b_\alpha^\dagger|n_1, n_2, \ldots, n_\alpha, \ldots\rangle \equiv (-)^{\sum_{\beta<\alpha} n_\beta}|n_1, n_2, \ldots, 1+n_\alpha, \ldots\rangle, \tag{10}$$

where it is understood that $|n_1, n_2, \ldots\rangle \equiv 0$ if any $n_\beta > 1$, implies that $\{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha\beta}$ and all other anticommutators are zero.

Solution: Apply $b_\beta^\dagger$ for $\beta \leq \alpha$ to the defining relation for $b_\alpha^\dagger$:

$$b_\beta^\dagger b_\alpha^\dagger|n_1, n_2, \ldots, n_\beta, \cdots n_\alpha, \ldots\rangle \equiv (-)^{\sum_{\beta\leq\gamma<\alpha} n_\gamma}|n_1, n_2, \ldots, 1+n_\beta, \ldots, 1+n_\alpha, \ldots\rangle, \tag{11}$$

If we applied $b_\beta^\dagger$ first there would be an extra minus multiplying the right side because the exponent 0f $(-)$ from the application of $b_\alpha^\dagger$ will have m_β increased by 1. It follows that $\{b_\beta^\dagger, b_\alpha^\dagger\} = 0$. By considering $\langle n_1, \ldots, n_\alpha, \ldots|b_\alpha|m_1, \ldots, m_\alpha, \ldots\rangle$ we infer that

$$b_\alpha|n_1, n_2, \ldots, n_\alpha, \ldots\rangle = (-)^{\sum_{\beta<\alpha} n_\beta}|n_1, n_2, \ldots, -1+n_\alpha, \ldots\rangle, \tag{12}$$

Then by the same argument, if $\alpha \neq \beta$ we find that $\{b_\alpha . b_\beta^\dagger\} = 0$. However if $\alpha = \beta$, then if $n_\alpha = 0$, $b_\alpha b_\alpha^\dagger$ leaves the state unchanged and $b_\alpha^\dagger b_\alpha$ applied to the state gives 0. If $n_\alpha = 1$ the reverse happens, so it follows that $\{b_\alpha . b_\alpha^\dagger\} = 1$.