Quantum Field Theory

Problem Set 5

Due: Friday, 15 February 2019

Reading: Lecture Notes, Chapters 2-3. S, Sections 33-36,40.

16. In problem 13 of Problem Set 4, you showed that the Lorentz transformation of the Dirac field is $\psi(x) \to e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\psi(\Lambda^{-1}x)$ and in class we learned that the parity transformation is $\psi(\boldsymbol{x}, t) \to e^{i\phi}\gamma^0\psi(-\boldsymbol{x}, t)$. Show from these rules that the Dirac bilinears, $\bar{\psi}(x)\psi(x)$, $\bar{\psi}(x)\gamma_5\psi(x)$, $\bar{\psi}(x)\gamma^\mu\psi(x)$, $\bar{\psi}(x)\gamma_5\gamma^\mu\psi(x)$, and $\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$, transform under Lorentz and parity transformations as scalar, pseudoscalar, vector, axial vector, and second rank tensor fields respectively. Here $\bar{\psi} \equiv \psi^\dagger\gamma^0$ is the Dirac adjoint. A useful first step in the solution is to prove that $\bar{\psi}(x) \to \bar{\psi}(\Lambda^{-1}x)e^{+i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}$ under Lorentz transformations. Also it is enough to specialize to infinitesimal Lorentz transformations. (The terms pseudo- and axial refer to opposite than "normal" parity properties. For a normal 4-vector, the time component is even and the space component is odd under parity. A normal scalar is even under parity.)

So;ution: Because $\sigma^{\mu\nu\dagger} = \gamma^0\sigma^{\mu\nu}\gamma^0$, $\bar{\psi}A\psi \to \bar{\psi}e^{+i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}Ae^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\psi$ for any matrix A. If $A = I$ or $A = \gamma_5$ the bilinear is a scalar under proper Lorentz transformations since both commute with $\sigma^{\mu\nu}$. If $A = \gamma^\mu$ or $A = \gamma_5\gamma^\mu$, problem 9b) above shows that the bilinear is a four vector under proper L.T.'s, and $\gamma^\mu\gamma^\nu$ (and hence also $\sigma^{\mu\nu}$) is a two index tensor under proper L.T.'s. Since the parity transformation on ψ includes multiplication by $\beta = \gamma^0$, and $\gamma_5\gamma^0 = -\gamma^0\gamma_5$, an extra γ_5 in the bilinear produces an extra $-$ under parity converting a proper tensor to a pseudo-tensor.

17. Show that the Dirac equation, regarded as a single particle Schrödinger equation, implies that the free electron has a Landé g factor of 2 ($\boldsymbol{\mu} \equiv g(Q/(2mc)\boldsymbol{S})$. [Hint: Identify the magnetic moment by examining the Dirac equation for a weak slowly varying magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ in the limit that the electron is moving slowly. We assume that the magnetic field enters the Dirac equation via the minimal substitution $\nabla \to \nabla - iq\boldsymbol{A}$. In this situation, after eliminating the lower two components of $\psi = \binom{\phi}{\chi}$ in favor of the upper two components in standard representation, the D.E. should go over to the NR Schrödinger equation with a term that can be identified as $-\boldsymbol{\mu}\cdot\boldsymbol{B}$.]

So;ution: For a slow electron it is convenient to write $\psi = e^{-imt}\psi_0$. Then time derivatives of ψ_0 can be neglected compared to m and we find $\chi_0 \approx \boldsymbol{\sigma}\cdot(\nabla - iq\boldsymbol{A})\phi_0$. Then we find the equation for ϕ_0:

$$i\dot{\phi}_0 \approx -\frac{1}{2m}[\boldsymbol{\sigma}\cdot(\nabla - iq\boldsymbol{A})]^2\phi_0 = -\frac{1}{2m}(\nabla - iq\boldsymbol{A})^2\phi_0 - \frac{q}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}\phi_0$$

where we used $\sigma^k\sigma^l = \delta_{kl} + i\epsilon^{klm}\sigma^m$. This equation has the form of the nonrelativistic Schrödinger equation with a spin contribution to the magnetic moment. The last term shows that the spin part of the magnetic moment is $\boldsymbol{\mu} = q\boldsymbol{\sigma}/2m = q\boldsymbol{S}/m \equiv gq\boldsymbol{S}/2m$. That is $g = 2$.

18. Take the Dirac equation for an electron in the Coulomb field of a proton to be

$$\left(i\gamma\cdot\partial+\gamma^0\frac{e^2}{4\pi r}-m\right)\psi=0.$$

(a) Writing the wave function for a stationary state in the form

$$\psi(x)=e^{-imt-iEt}\begin{pmatrix}\phi(\mathbf{x})\\ \chi(\mathbf{x})\end{pmatrix}$$

where ϕ and χ are 2 component spinors, solve for χ in terms of ϕ and show that ϕ satisfies the equation

$$\left\{-\boldsymbol{\sigma}\cdot\nabla\left(2m+E+\frac{e^2}{4\pi r}\right)^{-1}\boldsymbol{\sigma}\cdot\nabla-\frac{e^2}{4\pi r}\right\}\phi=E\phi.$$

So;ution: Writing D.E. in std rep, gives

$$(m+E)\phi=\left(m-\frac{e^2}{4\pi r}\right)\phi+\frac{1}{i}\vec{\sigma}\cdot\nabla\chi;\quad (m+E)\chi=\left(-m-\frac{e^2}{4\pi r}\right)\chi+\frac{1}{i}\vec{\sigma}\cdot\nabla\phi$$

Solve 2nd eq for χ, plug in first to get desired eq.

(b) By taking suitable limits on the Dirac equation in this form, find the first relativistic corrections to the nonrelativistic Schrödinger equation. Identify the correction terms with the familiar spin-orbit and relativistic kinetic energy correction terms used to understand the fine structure splittings in non-relativistic quantum mechanics, paying attention to any differences. What can you conclude about the splittings of the 2s and 2p levels of hydrogen?

[Hint: Expand the inverse in the first term on the l.h.s. Remember that in units with $\hbar=c=1$, $e^2/4\pi\approx 1/137<<1$, and the Bohr radius of hydrogen $\gg$ the Compton wavelength of the electron, so we may assume E and $e^2/4\pi r$ are both $<<m$.]

Solution: For zeroth order, $E,e^2/4\pi r=O(m\alpha^2)$ and $\nabla=O(m\alpha)$. Thus it is enough to keep just 2 terms in expansion $(2m+E+e^2/4\pi r)^{-1}\approx(1-E/2m-e^2/8\pi mr)/2m$, since the two ∇'s it multiplies are $O(m^2\alpha^2)$. For the same reason, we can replace $(E+e^2/4\pi r)\phi\to-(\nabla^2/2m)\phi$ in that term. To do this, we have to bring the $1/r$ next to the ϕ by using $[1/r,\vec{\sigma}\cdot\nabla]=\vec{r}\cdot\vec{\sigma}/r^3$. After all these changes the eq reads

$$\left\{-\frac{\nabla^2}{2m}-\frac{e^2}{4\pi r}\right\}\phi+\frac{1}{4m^2}\left\{\vec{\sigma}\cdot\nabla\frac{e^2\vec{r}\cdot\vec{\sigma}}{4\pi r^3}-\frac{\nabla^4}{2m}\right\}\phi=E\phi.$$

Next use $\sigma^k\sigma^l=\delta_{kl}+i\epsilon^{klm}\sigma^m$ to simplify the terms involving $\vec{\sigma}$

$$\left\{-\frac{\nabla^2}{2m}-\frac{e^2}{4\pi r}\right\}\phi+\frac{1}{4m^2}\left\{\frac{e^2\vec{r}}{4\pi r^3}\cdot\nabla+e^2\delta(\vec{r})+\frac{e^2\vec{\sigma}\cdot\vec{L}}{4\pi r^3}-\frac{\nabla^4}{2m}\right\}\phi=E\phi.$$

where $\vec{L} = -i\vec{r} \times \nabla$ is the orbital angular momentum. Finally we have to consider the normalization condition of the Dirac wavefunction

$$1 = \int d^3x(\phi^\dagger\phi + \chi^\dagger\chi) \approx \int d^3x\phi^\dagger \left(1 - \frac{\nabla^2}{4m^2}\right)\phi$$

which shows that the Schrodinger w.f. should be identified with $\Psi_S = \sqrt{1 - \frac{\nabla^2}{4m^2}}\phi \approx \left(1 - \frac{\nabla^2}{8m^2}\right)\phi$. To convert ϕ to Ψ_S in the equation we apply $(1 - \nabla^2/8m^2)$ to both sides. We pick up the commutator $[-\nabla^2, 1/r] = 4\pi\delta(\vec{r}) + 2(\vec{r}/r^3)\cdot\nabla$ in the first term. (The commutators from the second term are higher order than we keep).

$$\left\{-\frac{\nabla^2}{2m} - \frac{e^2}{4\pi r} + \frac{e^2}{8m^2}\delta(\vec{r}) + \frac{e^2\vec{\sigma}\cdot\vec{L}}{16m^2\pi r^3} - \frac{\nabla^4}{8m^3}\right\}\Psi_S = E\Psi_S.$$

The last three terms in the braces show the relativistic corrections. The fourth term is the standard spin orbit coupling, the fifth term is the standard relativistic correction to the kinetic energy, and the third term is the so-called Darwin term, which contributes only to $l = 0$ level shifts. Thus for $l \neq 0$ we read off the level shifts from any text on quantum mechanics:

$$\Delta E_{njl} = -\frac{mc^2\alpha^4}{2n^3}\left[\frac{1}{j + 1/2} - \frac{3}{4n}\right], \qquad l = j \pm 1/2 > 0$$

For $l = 0$ ($j = 1/2$) the spin orbit term vanishes but we have the Darwin shift $\alpha\pi R_{n0}^2(0)/2m^2 = mc^2\alpha^4/2n^3$ added to the shift from the K.E. correction, and the result is the above for $j = 1/2$. Thus the D.E. equation predicts corrections that depend only on n and the total angular momentum j. In particular, the $2s_{1/2}$ and $2p_{1/2}$ levels remain degenerate. Experimentally they are not degenerate, split by the famous Lamb shift, which is caused by QED radiative corrections.

19. **Quantum Numbers of Positronium.**

a)A state of positronium (*i.e.* a hydrogenic e^+e^- atom) can be written as

$$|\Psi\rangle = \int d^3p \sum_{\mu_1\mu_2} F(\vec{p}, \mu_1; -\vec{p}, \mu_2) b^\dagger_{\vec{p}\mu_1} d^\dagger_{-\vec{p}\mu_2}|0\rangle.$$

where it is convenient to label spin not by helicity but by the spinor basis $\phi_\mu = \binom{1}{0}, \binom{0}{1}$ which are just eigenstates of σ^z, so that parity does not touch spin. When F corresponds to orbital angular momentum L and total spin $S (S = 0$ or $1)$ how does $|\Psi\rangle$ transform under parity and charge conjugation?

So;ution: The positron and electron have negative relative intrinsic parity. Combining that with the parity $(-)^L$ associated with orbital angular momentum gives combined parity $(-)^{L+1}$. Thus the parity of S, P, D wave functions is $-, +, -$ respectively. Charge conjugation changes the electron and positron into each other. That is equivalent to exchanging the spin $(-)^{S+1}$ and orbital labels $(-)^L$ of the two particles plus an extra sign because $\{b^\dagger, d^\dagger\} = 0$: $C = (-)^{L+S}$. For triplet S, P, D this is $-, +, -$ and for singlet S, P, D it is $+, -, +$.

(b) It is a fact that an n photon state has charge conjugation eigenvalue $C = (-)^n$. If C is conserved in the spontaneous annihilation of positronium into photons, what is the minimum number of photons in the final state of the decay of the ground states of ortho-(S=1) and para-(S=0) positronium?

So;ution: Since charge conjugation reverses the charge of a particle, we should have $Cj^\mu C^{-1} = -j^\mu$ (which you have checked in a previous problem). Thus $A_\mu \to -A_\mu$ under charge conjugation will make charge conjugation a symmetry of QED. This implies that a photon is odd under C and an n photon state has $C = (-)^n$. States of positronium with $L + S$ even have $C = +$ and hence annihilate only into an even number of photons. This includes the spin-0 ground state. States with $L + S$ odd, which include the spin-1 ground state must decay into an odd number of photons. Energy and momentum can't be conserved in the annihilation of positronium into a single photon, so the minimum number of photons from the decay of orthopositronium is 3. More generally, one photon transitions must change C, so they can only occur between a state with $L + S$ even and a state with $L + S$ odd.