Quantum Field Theory

Problem Set 6

Due:Friday, 22 February 2019

Reading:Lecture Notes, Chapter 3,4,5.

20. Revisit our definition of the charge conjugation transformation of the Dirac field,, but this time using a Majorana representation where all the γ^μ are imaginary. Show that in this representation there is no need for a matrix like $i\gamma^2$ that we needed in the standard representation. Also confirm that the Chiral representation does require this matrix.

Solution: We first note that the u, v spinors satisfy Dirac equations

$$(\gamma \cdot p + m)u = 0, \qquad (\gamma \cdot p - m)v = 0 \tag{1}$$

respectively. With all four gammas imaginary, if equation then $v = u^*$ satisfies the second. If the γ's are imaginary, so are the $\sigma^{\mu\nu}$, so if u has helicity λ, then v will have helicity $-\lambda$. Thus $v = u^*$ satisfies the equations that define it. Using u, v in the expasnion of the field then shows that the charge conjugation transformation is $\psi \rightarrow 9\psi^\dagger)^T$ in a Majorana representaton.

21. As mentioned in class, any system of spin 1/2 fields can be described in terms of a collection of Weyl fields with the same chirality, say $L_k(x)$. Then the most general mass term can be expressed in the form

$$\sum_{kl} \frac{m_{kl}}{2} L_k^T i\gamma^2 \beta L_l + \sum_{kl} \left(\frac{m_{kl}}{2} L_k^T i\gamma^2 \beta L_l \right)^\dagger . \tag{2}$$

where the complex matrix m is symmetric $m^T = m$. A unitary transformation $L_k \rightarrow U_{km} L_m$ leaves the kinetic term $\int d^3x \sum_k L_k^\dagger (-i\boldsymbol{\alpha} \cdot \nabla) L_k$ invariant.

a) Show that the change of variables $L = UL'$, with $U^\dagger U = I$, modifies the mass matrix to $m' = U^T m U$. Note that this is *not* a similarity transformation when U is complex!

Solution: Inserting $U_{km} L_m$ for L_k, we find

$$\sum_{kl} \frac{m_{kl}}{2} (UL)_k^T i\gamma^2 \beta (UL)_l = \sum_{kl} \frac{m_{kl}}{2} (L^T U^T)_k i\gamma^2 \beta (UL)_l = \sum_{klmn} U_{mk}^T \frac{m_{kl}}{2} U_{ln} L_m^T i\gamma^2 \beta L_n \tag{3}$$

showing that the new mass matrix is $U^T m U$.

b) Show that one can choose U so that m' is a diagonal matrix $m'_{kl} = m_k \delta_{kl}$. *Hint*: first note that there is a unitary V which diagonalizes the hermitian matrix $m^\dagger m$. Then prove that the real and imaginary parts of $V^T m V$ commute with each other, and so can be simultaneously diagonalized by a real orthogonal matrix similarity transformation..

Solution: The matrix m is symmetric and complex, so it is neither real symmetric nor hermitian. However the matrix $m^\dagger m$ *is* Hermitian, so we know from basic quantum mechanics that there is a unitary V such that $V^\dagger m^\dagger m V$ is diagonal, and moreover with non negative real diagonal entries. Next consider the symmetric matrix $V^T m V = X + iY$, where X, Y are respectively the real and imaginary parts of m. Then $V^\dagger m^\dagger V^* = X - iY$. Then we compute

$$(X - iY)(X + iY) = V^\dagger m^\dagger V^* V^T m V = V^\dagger m^\dagger m V \tag{4}$$

But V was chosen so that the right side is real and diagonal. Therefore the imaginary part of the left side is zero:

$$\mathrm{Im}(X - iY)(X + iY) = \mathrm{Im}(X^2 + Y^2 + i[X, Y]) = [X, Y] = 0 \tag{5}$$

Thus X and Y can be simultaneously diagonalized by a real orthogonal matrix R, $R^T(X + iY)R = R^T V^T m V R$ is diagonal, which was to be proved, since VR is a unitary matrix. Note that the diagonal entries need not be real. However if we write the diagonal entries $\mu_k = \rho_k e^{i\phi_k}$ the phases can be removed by one last transformation given by the diagonal unitary matrix $\mathrm{diag}(e^{i\phi_k/2})$.

c) In the case of 2 fields L_k, $k = 1, 2$, apply this procedure to diagonalize the mass matrix $\begin{pmatrix} 0 & m \\ m & M \end{pmatrix}$, where m, M are both allowed to be complex. Discuss what happens to the mass eigenvalues when $|M| \gg |m|$. *HInt*: Start by multiplying the mass matrix on the left and right by a diagonal matrix with entries $e^{i\alpha}, e^{i\beta}$ and choose α, β so the resulting matrix is real, and hence hermitian.

Solution: Write $m = |m|e^{i\phi}$ and $M = |M|e^{i\Phi}$. Then multiplying the mass matrix on the left and right by the diagonal matrix with entries $e^{i\alpha}, e^{i\beta}$ we find

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} \begin{pmatrix} 0 & m \\ m & M \end{pmatrix} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} = \begin{pmatrix} 0 & m e^{i(\alpha+\beta)} \\ m e^{i(\alpha+\beta)} & M e^{i2\beta} \end{pmatrix} \tag{6}$$

Then we choose $\beta = -\Phi/2$ and $\alpha + \beta = -\phi$ leaves us with a real symmetric matrix that can be diagonalized by a real orthogonal matrix R, $R^T R = I$, which is also unitary. The characteristic equation is then

$$x(x - |M|) - |m|^2 = 0, \qquad x^\pm = \frac{|M| \pm \sqrt{|M|^2 + 4|m|^2}}{2} \tag{7}$$

When $|M| \gg |m|$, $m^+ = |M| + O(m^2/M)$ and $m^- = -|m|^2/m^+ \sim -|m|^2/|M|$. This type of mass matrix would explain a very light neutrino mass compared to the charged particle masses, even with m the same size as the charged leptons. This is the seesaw mechanism suggested by Gell-Mann, Ramond, Slansky, and Fritsch in the late 70's.

22. **Time reversal and charge conjugation**

a) Derive how the Dirac bilinears, $\bar{\psi}(x)\psi(x)$, $\bar{\psi}(x)i\gamma_5\psi(x)$, $\bar{\psi}(x)\gamma^\mu\psi(x)$, $\bar{\psi}(x)\gamma_5\gamma^\mu\psi(x)$, and $\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$ transform under time reversal when ψ is the second quantized Dirac field operator.

Solution: Under time reversal $\psi(\vec{x},t) \to i\Sigma_2\psi(\vec{x},-t)$, and any extra c-numbers are complex conjugated. Thus $\bar{\psi}A\psi \to \bar{\psi}(-i\Sigma_2)A^* i\Sigma_2\psi$, where $t \to -t$ is understood. So all we need do is work out $\Sigma_2 A^*\Sigma_2$ for all cases: $\Sigma_2 I^*\Sigma_2 = I$, $\Sigma_2(i\gamma_5)^*\Sigma_2 = -i\gamma_5$, $\Sigma_2\gamma^{0*}\Sigma_2 = \gamma^0$, $\Sigma_2\gamma^{k*}\Sigma_2 = -\gamma^k$ $\Sigma_2\gamma_5^*\gamma^{0*}\Sigma_2 = \gamma_5\gamma^0$, $\Sigma_2\gamma_5^*\gamma^{k*}\Sigma_2 = -\gamma_5\gamma^k$ $\Sigma_2\sigma^{0i*}\Sigma_2 = +\sigma^{0i}$, $\Sigma_2\sigma^{ij*}\Sigma_2 = -\sigma^{ij}$. In summary for the 5 basic bilinears the multiplicative sign associated with time reversal is,respectively, $(+,-,+,+,-)(-)^S$ where S in the number of *spatial* indices.

(b) How do these same bilinears transform under charge conjugation.

Solution Charge conjugation is $\psi \to i\gamma^2(\psi^\dagger)^T$. Then $\bar{\psi}A\psi \to \psi^T i\gamma^2\gamma^0 A i\gamma^2(\psi^\dagger)^T = -\psi^\dagger i\gamma^2 A^T\gamma^0 i\gamma^2\psi = -\bar{\psi}\gamma^0 i\gamma^2 A^T\gamma^0 i\gamma^2\psi$. We find $-\gamma^0 i\gamma^2 I^T\gamma^0 i\gamma^2 = I$, $-\gamma^0 i\gamma^2(i\gamma_5)^T\gamma^0 i\gamma^2 = i\gamma_5$, $-\gamma^0 i\gamma^2(\gamma^\mu)^T\gamma^0 i\gamma^2 = -\gamma^\mu$, $-\gamma^0 i\gamma^2(\gamma_5\gamma^\mu)^T\gamma^0 i\gamma^2 = +\gamma_5\gamma^\mu$, $-\gamma^0 i\gamma^2(\sigma^{\mu\nu})^T\gamma^0 i\gamma^2 = -\sigma^{\mu\nu}$. In summary we have the signs $(+,+,-,+,-)$ for charge conjugation.

23. Combine the transformations of parity P charge conjugation C and time reversal on the bilinears, which you derived in problem 16 (from set 5) and in problem 22, in all possible ways, CP, CT, PT, and CPT, and detemine how the various bilinears transform under each combination. Thus you will confirm the results quoted in section 3.6 of our Lecture Notes.

Solution: We first summarize the transformation properties of the bilinears found in the prefious exercises. $\bar{\psi}_A(x)\Gamma_k\psi_B(x)$, where the matrices Γ_k are in turn $(I, i\gamma_5, \gamma^\mu, \gamma_5\gamma^\mu, \sigma^{\mu\nu}$. We found:

$$\begin{aligned}
P^{-1}\bar{\psi}_A(\boldsymbol{x},t)\Gamma_k\psi_B(\boldsymbol{x},t)P &= (-)^S(+,-,+,-,+)\bar{\psi}_A(-\boldsymbol{x},t)\Gamma_k\psi_B(-\boldsymbol{x},t)\\
C^{-1}\bar{\psi}_A(\boldsymbol{x},t)\Gamma_k\psi_B(\boldsymbol{x},t)C &= (+,+,-,+,-)\bar{\psi}_B(\boldsymbol{x},t)\Gamma_k\psi_A(\boldsymbol{x},t)\\
T^{-1}\bar{\psi}_A(\boldsymbol{x},t)\Gamma_k\psi_B(\boldsymbol{x},t)T &= (-)^S(+,-,+,+,-)(-)^S\bar{\psi}_A(\boldsymbol{x},-t)\Gamma_k\psi_B(\boldsymbol{x},-t)
\end{aligned}$$

First apply parity to the second equation:

$$\begin{aligned}
(CP)^{-1}\bar{\psi}_A(\boldsymbol{x},t)\Gamma_k\psi_B(\boldsymbol{x},t)CP &= (-)^S(+,-,+,-,+)(+,+,-,+,-)\bar{\psi}_B(-\boldsymbol{x},t)\Gamma_k\psi_A(-\boldsymbol{x},t)\\
&= (-)^S(+,-,-,-,-)\bar{\psi}_B(-\boldsymbol{x},t)\Gamma_k\psi_A(-\boldsymbol{x},t)
\end{aligned}$$

Applying time reversal to this result gives

$$\begin{aligned}
(CPT)^{-1}\bar{\psi}_A(\boldsymbol{x},t)\Gamma_k\psi_B(\boldsymbol{x},t)CPT &= (+,+,-,-,+)\bar{\psi}_B(-\boldsymbol{x},-t)\Gamma_k\psi_A(-\boldsymbol{x},-t)\\
&= (-)^{n_\Gamma}\bar{\psi}_B(-x)\Gamma_k\psi_A(-x) = (-)^{n_\Gamma}(\bar{\psi}_A(-x)\Gamma_k\psi_B(-x))^\dagger
\end{aligned}$$

as found in Section 3.6.

To get PT and CT we apply time reversal to the first and second equations respectively:

$$\begin{aligned}
(PT)^{-1}\bar{\psi}_A(\boldsymbol{x},t)\Gamma_k\psi_B(\boldsymbol{x},t)PT &= (+,+,+,-,-)\bar{\psi}_A(-\boldsymbol{x},-t)\Gamma_k\psi_B(-\boldsymbol{x},-t)\\
(CT)^{-1}\bar{\psi}_A(\boldsymbol{x},t)\Gamma_k\psi_B(\boldsymbol{x},t)CT &= (-)^S(+,-,-,+,+)\bar{\psi}_B(\boldsymbol{x},-t)\Gamma_k\psi_A(\boldsymbol{x},-t)
\end{aligned}$$