Quantum Field Theory

Solution Set 7

Due: 1 March 2019

Reading: Lecture Notes, Chapters 4-5

24. The results of problem 16 (Problem Set 5) are based on the fact that the similarity transformation

$$e^{i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\Gamma e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4} \tag{1}$$

with Γ any of the matrices $I, i\gamma_5, \gamma^\mu, \gamma_5\gamma^\mu, \sigma^{\mu\nu}$ performs a Lorentz transformation on each four-vector index. As noted in part a) below, the matrices

$$e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4} \tag{2}$$

give us the (nonunitary) representation $D(1/2,0)\oplus D(0,1/2)$ of the Lorentz group.

a) Show that the Lorentz generator matrices $M^{\mu\nu} = (i/4)[\gamma^\mu,\gamma^\nu]$, in the natural (chiral) representation, take the block diagonal forms

$$M^{kl} = \frac{1}{2}\epsilon_{klm}\begin{pmatrix}\sigma^m & 0\\ 0 & \sigma^m\end{pmatrix} \tag{3}$$

$$M^{0k} = \frac{i}{2}\begin{pmatrix}-\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix} \tag{4}$$

This shows that the Lorentz group has two distinct two dimensional representations called $D(1/2,0)$ represented by the upper 2×2 block and $D(0,1/2)$ represented by the lower 2×2 block. The Dirac representation is reducible into the direct sum of these.

Solution: In the chiral representation the spatial gammas are identical to the standard representation, but γ^0 is replaced by $\gamma^0 = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$. Then by direct calculation we have

$$\begin{aligned}
\gamma^0\gamma^k &= \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^k\\ -\sigma^k & 0\end{pmatrix} = \begin{pmatrix}-\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}\\
M^{0k} &= \frac{i}{4}[\gamma^0,\gamma^k] = \frac{i}{2}\begin{pmatrix}-\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix} \qquad (5)\\
M^{kl} &= \frac{i}{4}[\gamma^k,\gamma^l] = \frac{i}{4}\begin{pmatrix}-[\sigma^k,\sigma^l] & 0\\ 0 & -[\sigma^k,\sigma^l]\end{pmatrix} = \frac{i}{4}\epsilon_{klm}\begin{pmatrix}-2i\sigma^m & 0\\ 0 & -2i\sigma^m\end{pmatrix}\\
&= \frac{i}{2}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix} \qquad (6)
\end{aligned}$$

as desired.

b) Show that the matrices

$$e^{i\lambda_{\mu\nu}\sigma^{T\mu\nu}/4}$$

are similar to the matrices (2). (Recall that a similarity transformation on a matrix A is $S^{-1}AS$ for an invertible matrix S. Simply find an S that does the trick.)

Solution: It is enough to show that $\sigma^{T\mu\nu}$ is similar to$-\sigma^{\mu\nu}$. First note that $\gamma^{\mu *} = -i\gamma^2\gamma^\mu i\gamma^2$ and hence $\gamma^{T\mu} = -i\gamma^2\gamma^0\gamma^\mu\gamma^0 i\gamma^2$, so $\gamma^{\mu T}$ is similar to $-\gamma^\mu$. It follows that $\sigma^{T\mu\nu}$ is similar to $\sigma^{\nu\mu} = -\sigma^{\mu\nu}$.

25. From the previous problem the transformation (1) may be viewed as belonging to the

$$(D(1/2,0) \oplus D(0,1/2)) \otimes (D(1/2,0) \oplus D(0,1/2)) \tag{7}$$

representation of the Lorentz group: Think of the matrix elements Γ_{ab} as a two-index bi-spinor, for which the Lorentz transformation reads

$$\Gamma'_{ab} = \Gamma_{cd}(e^{i\lambda_{\mu\nu}\sigma^{T\mu\nu}/4})_{ca}(e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4})_{db}$$

a) Find the decomposition of the tensor product representation (7) into irreducible representations of the Lorentz group. Note that the decomposition follows the same rules as for $SU(2)$, namely

$$\frac{1}{2} \otimes 0 = \frac{1}{2}, \qquad \frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0.$$

Solution:

$$\begin{aligned}
&(D(1/2,0) \oplus D(0,1/2)) \otimes (D(1/2,0) \oplus D(0,1/2)) \\
&= (D(1/2,0) \otimes D(1/2,0)) \oplus (D(0,1/2) \otimes D(0,1/2)) \\
&\oplus (D(1/2,0) \otimes D(0,1/2)) \oplus (D(0,1/2) \otimes D(1/2,0) \\
&= D(1,0) \oplus D(0,0) \oplus D(0,1) \oplus D(0,0) \oplus D(1/2,1/2) \oplus D(1/2,1/2)
\end{aligned}$$

b) By considering the dimensionalities of the representations, relate the results of part (b) to the transformation properties of the matrices $I, i\gamma_5, \gamma^\mu, \gamma_5\gamma^\mu, \sigma^{\mu\nu}$.

Solution: $D(1/2,1/2)$ is the four vector representation of the Lorentz group and it occurs twice here: they must correspond to γ^μ and $\gamma_5\gamma^\mu$. The scalar representation $D(0,0)$ also occurs twice corresponding to $i\gamma_5$ and I. Finally $D(1,0) \oplus D(0,1)$ is a six dimensional representation which by the process of elimination must correspond to $\sigma^{\mu\nu}$, the antisymmetric rank two tensor representation. (note that the dimensions of these representations match.)

26. In class and also as you have found in the previous problem, the irreducible representation $D(1/2,1/2)$ of the Lorentz group is identified with the 4-vector representation $p^\mu \to \Lambda^\mu{}_\nu p^\nu$. A concrete way to understand this result is in terms of the 2×2 matrix $P \equiv p^0 I + \boldsymbol{p} \cdot \boldsymbol{\sigma}$. Note that P is Hermitian $P^\dagger = P$ when p^μ is a real 4-vector.

a) Show that $\det P = (p^0)^2 - \boldsymbol{p}^2 = -\eta_{\mu\nu}p^\mu p^\nu$. This shows that a transformation on the matrix P which leaves $\det P$ invariant must induce a four vector Lorentz transformation on p^μ.

Solution: Writing out P explicitly

$$P = \begin{pmatrix} p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & p^0-p^3 \end{pmatrix}, \tag{8}$$

we immediately see that $\det P = (p^0)^2 - (p^3)^2 - ((p^1)^2 + (p^2)^2) = -\eta_{\mu\nu}p^\mu p^\nu = -p \cdot p$.

b) To keep P hermitian, we should restrict the transformations of P to the form $P \to L^\dagger P L$, where L is a 2×2 matrix. Then this transformation leaves $\det P$ invariant if $|\det L| = 1$. Multiplying L by a phase has no effect on $L^\dagger P L$ so wolog we can take $\det L = 1$, so L is in the group of matrices $SL(2, C)$, the group of complex 2×2 matrices with unit determinant. Show that the matrices representing $D(1/2, 0)$ and $D(0, 1/2)$ have unit determinant, and complete the argument that $D(1/2, 1/2)$ is the 4-vector representation of the Lorentz group.

Solution: As we have seen the rotation and boost generators in these representations are $\sigma/2$ and $\mp i\sigma/2$. Since these matrices have zero trace, the determinants of the group elements which are exponentials of the generators are unity in both cases. To complete the argument, interpret the transformation $L^\dagger P L$ as a bispinor representation

$$(L^\dagger P L)_{ab} = L^*_{ca} P_{cd} L_{db} = P_{cd}[L^*_{ca} L_{db}] \tag{9}$$

which shows that the transformation is in the tensor product $D(1/2, 0) \otimes D(0, 1/2) = D(1/2, 1/2)$. But from part a) this transformation induces a 4-vector transformation on p^μ which is what we wanted to show.

27. Decompose the representation $D(1/2, 1/2) \otimes D(1/2, 1/2)$ into irreducible representations and, with the result of the previous problem in mind, interpret the individual components in terms of two index 4-tensors of different symmetry types.

Solution: The irreducible content of this tensor product can ve infrerred by regarding it as the tensor product two $D(1/2, 0)$ and two $D(0, 1/2)$:

$$(D(1, 0) \oplus D(0, 0)) \otimes (D(0, 1) \oplus D(0, 0)) = D(1, 1) \oplus D(0, 0) \oplus D(1, 0) \oplus D(0, 1) \tag{10}$$

On the other hand we know that the tensor product of two four vectors is a two index tensor with 16 components which can be divided into an antisymmetric tensor with 6 components, a traceless symmetric tensor with 10-1=9 components, and the scalar trace: 16=6+9+1. Comparing to our decomposition, we can infer that $D(1, 1)$ which has $3 \times 3 = 9$ components must be the traceless symmetric tensor, that $D(0, 0)$ corresponds to the scalar and by the process of elimination that the $3 + 3 = 6$ components of $D(1, 0) \oplus D(0, 1)$ must correpond to the antisymmetric tensor. One can isolate the irreducible components of an antisymmetric tensor as self dual and anti dual satisfyung

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma} = \pm i A^{\mu\nu} \tag{11}$$

The i in the duality relations signifies that the representations $D(1,0)$ and $D(0,1)$ are complex.