Quantum Field Theory

Solution Set 8

Due: 15 March 2019

Reading: Lecture Notes, Chapters 5-6

28. We have taken it for granted that the evolution operator $U(t_2, t_1) = T\exp\left\{-i\int_{t_1}^{t_2} H(t)dt\right\}$ is unitary if $H(t)$ is Hermitian.

a) Prove this by using the differential equation satisfied by $U(t, t_1)$.

Solution: $i\dot{U}(t, t_1) = H(t)U(t, t_1)$ implies $-i\dot{U}^\dagger = U^\dagger H^\dagger(t) = U^\dagger H(t)$ so $d(U^\dagger U)/dt = U^\dagger(H(t) - H(t))U = 0$. Then $U(t_1, t_1) = I$ implies $U^\dagger U = I$ for all t.

b) The result must also follow by expanding U and $U^\dagger$ in power series and collecting together all terms with N $H(t)$'s multiplied together. For each $N > 0$ these terms must cancel to give zero. Check this for $N = 1, 2$. [The ambitious can try to construct a proof for general N but do not waste time on this. It is simple if you hit on the right way, very nasty otherwise.]

Solution: The general argument first: The expansion of U involves time ordered products of H, whereas the expansion of $U^\dagger$ involves anti-time ordered products of H. When we expand $U^\dagger U$ and collect together all terms with N $H(t)$'s multiplied together, we find $N + 1$ terms $H(t_N)H(t_{N-1})\cdots H(t_1)$, each term with different limits on the ranges of the t_i as follows: $t_N < t_{N-1} < \cdots < t_2 < t_1$; $t_N < t_{N-1} < \cdots < t_2, t_1$; $t_N < t_{N-1} < \cdots < t_3, t_2 > t_1$; $t_N < t_{N-1} < \cdots < t_4, t_3 > t_2 > t_1$; ... $t_N > t_{N-1} > \cdots > t_2 > t_1$. Here the commas in each set of inequalities separate groups of t_i's that range independently of each other. The coefficient of each term is ± 1 where the sign alternates from one term to the next. Thus inspecting the first two ranges we see that the first range removes the part of the second range where $t_2 < t_1$ and converts it to the range $t_N < t_{N-1} < \cdots < t_3 < t_2 > t_1$. Combining this range with the third range, the latter becomes $t_N < t_{N-1} < \cdots < t_3 > t_2 > t_1$. This process continues until the next to the last range, $t_N, t_{N-1} > \cdots > t_2 > t_1$, is converted to $t_N > t_{N-1} > \cdots > t_2 > t_1$, which then cancels the last range.

You were asked to just do the cases $N = 1$ and $N = 2$. $N = 1$ is trivial. Explicitly for $N = 2$, the three ranges are $t_2 < t_1$; t_2, t_1 unrestricted, and $t_2 > t_1$. Combining the first two ranges gives $t_2 > t_1$ which cancels the third range.

29. The frame dependence of the concept of simultaneity raises the issue of whether the concept of time ordering can be compatible with Lorentz invariance: two events can have opposite temporal ordering in different Lorentz frames.

a) Quantum fields are supposed to obey Einstein causality, namely $[A(x), B(y)] = 0$ for $x-y$ spacelike ($(x-y)^2 > 0$). Otherwise spacelike separated experiments could interfere with each other. Argue that, as a consequence of this causality, time ordred products of local field operators are compatible with Lorentz invariance.

Solution: According to the properties of Lorentz transformations, the time ordering of events separated by a timelike interval is the same in all frames. This is simply because an observer traveling at a speed less than that of light can perceive both events, So the frame dependence of the ordering can only occur when the two events are space-like separated. But for local operators the operators commute, so in that case $A(t_1)B(t_2) = B(t_2)A(t_1)$, so the frame dependence of time ordering gives the same result for the time ordered product in each frame.

b) Show that for field operators $A(x), B(y)$, that

$$\frac{\partial}{\partial t} T[A(\boldsymbol{x}, t)B(\boldsymbol{y}, t')] = T[\dot{A}(\boldsymbol{x}, t)B(\boldsymbol{y}, t')] + \delta(t' - t)[A(\boldsymbol{x}, t), B(\boldsymbol{y}, t)], \qquad \dot{A} \equiv \frac{\partial A}{\partial t} \quad (1)$$

if A or B is bosonic. If both A and B are fermionic the commutator in the last term is replaced by an anticommutator. This relation is important in resolving other apparent violations of Lorentz invariance in time dependent perturbation theory.

Solution: We can express the time ordered product symbolically using the step function $\theta(t) = 1$ for $t > 0$ and $= 0$ otherwise:

$$T[A(x)B(y)] = \theta(t - t')A(x)B(y) \pm \theta(t' - t)B(t')A(t) \quad (2)$$

where the minus is chosen if both A and B are fermionic. Now $d\theta(t)/dt = \delta(t)$. So taking the time derivative of both sides gives

$$\begin{aligned} \frac{d}{dt} T[A(x)B(y)] &= \theta(t - t')\dot{A}(x)B(y) \pm \theta(t' - t)B(t')\dot{A}(t) \\ &\quad + \delta(t - t')A(x)B(y) \mp \delta(t' - T)B(y)A(x) \end{aligned} \quad (3)$$

as desired.

30. Using the Hamiltonian of problem 8 of set 2, calculate the scattering amplitude for a single charged scalar particle to first order in a static 4 vector potential $A_\mu(\boldsymbol{x})$. Specialize to the Coulomb potential $\boldsymbol{A} = 0$, $A^0 = e/(4\pi|\boldsymbol{x}|)$, and calculate the differential cross section for this process as a function of the initial particle energy and scattering angle. [Since you are working only to first order in A_μ, the amplitude will be proportional to the matrix element of the current operator with $A_\mu = 0$

$$j_\mu(x) \approx -iq(\phi^\dagger \partial_\mu \phi - (\partial_\mu \phi^\dagger)\phi). \quad (4)$$

Also assume that j^μ is normal ordered so that its vacuum expectation value is zero.]

Solution: The interaction part of the hamiltonian is in interaction picture

$$H' = \int d^3x[iqA^\mu(\phi_I^\dagger\partial_\mu\phi_I - \partial_\mu\phi_I^\dagger\phi_I) + q^2\boldsymbol{A}^2\phi_I^\dagger\phi_I]$$

Here we used $\pi_I = \dot{\phi}_I^\dagger$ because interaction picture fields satisfy the zero field equations of motion. Note that the term quadratic in A involves spatial components only, but is second order in the field and so will not contribute to the first order amplitude. According to the Dyson formula the scattering amplitude is to first order

$$(-i)\int d^4x\langle 0|a_I(\boldsymbol{p}')iqA^\mu(\phi_I^\dagger\partial_\mu\phi_I - \partial_\mu\phi_I^\dagger\phi_I)a_I^\dagger(\boldsymbol{p})|0\rangle = \frac{q\tilde{A}_\mu(p'-p)i(p+p')^\mu}{(2\pi)^3\sqrt{4\omega\omega'}}.$$

Here $\tilde{A}_\mu(k) = \int d^4xe^{-ik\cdot x}A_\mu(x) = 2\pi\delta(k^0)\int d^3xe^{-i\boldsymbol{k}\cdot\boldsymbol{x}}A_\mu(\boldsymbol{x}) \equiv 2\pi\delta(k^0)\tilde{A}_\mu(\boldsymbol{k})$ for static fields. Thus the Feynman amplitude for this process is

$$\mathcal{M} = q\tilde{A}_\mu(\boldsymbol{p}-\boldsymbol{p}')i(p+p')^\mu \to -iq(\omega+\omega')\tilde{A}^0(\boldsymbol{p}-\boldsymbol{p}') = -2iq\omega\tilde{A}^0(\boldsymbol{p}-\boldsymbol{p}')$$

for the Coulomb potential using energy conservation.

$$\tilde{A}^0(\boldsymbol{k}) = \int d^3x\frac{Q}{4\pi|\boldsymbol{x}|}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = Q\frac{2\pi}{4\pi}\int_0^\infty rdr\frac{2i\sin(kr)}{ikr} = \frac{Q}{\boldsymbol{k}^2}$$

Putting everything together we have

$$d\sigma = \frac{d^3p'}{(2\pi)^3 2\omega'}\frac{1}{2\omega v}2\pi\delta(\omega'-\omega)4\omega^2\frac{q^2Q^2}{(\boldsymbol{p}-\boldsymbol{p}')^4} = d\Omega\frac{q^2Q^2\omega^2}{4\pi^2(\boldsymbol{p}-\boldsymbol{p}')^4}$$

Now $(\boldsymbol{p}-\boldsymbol{p}')^2 = 2\boldsymbol{p}^2(1-\cos\theta) = 4\boldsymbol{p}^2\sin^2(\theta/2)$ so finally

$$\frac{d\sigma}{d\Omega} = \left(\frac{qQ}{8\pi}\right)^2\frac{m^2+\boldsymbol{p}^2}{\boldsymbol{p}^4\sin^4(\theta/2)}$$

31. **Electron scattering in a purely magnetic field.**

(a) Show that for any purely magnetic field, the scattering calculated to lowest order does not alter the helicity.

Solution: The lowest order scattering amplitude is proportional to

$$\tilde{\boldsymbol{A}}\cdot\bar{u}'\boldsymbol{\gamma}u = \tilde{\boldsymbol{A}}\cdot u'^\dagger\boldsymbol{\alpha}u$$

. In std rep

$$\boldsymbol{\alpha} = \begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}, \qquad u_\lambda = \sqrt{m+\omega}\begin{pmatrix}\chi_\lambda\\ 2\lambda|\boldsymbol{p}|\chi_\lambda/(m+\omega)\end{pmatrix}$$

Thus since energy is conserved, $\omega' = \omega$, $|\boldsymbol{p}'| = |\boldsymbol{p}|$, and

$$u'^\dagger\boldsymbol{\alpha}u = 2(\lambda+\lambda')|\boldsymbol{p}|\chi_{\lambda'}'^\dagger\boldsymbol{\sigma}\chi_\lambda$$

which vanishes if $\lambda' = -\lambda$.

(b) Calculate the differential cross-section for electron scattering in the field of a magnetic dipole when the initial and final momenta are both perpendicular to the dipole axis, to lowest order in perturbation theory (Born approximation). [The vector potential of a dipole $\boldsymbol{M}$ is $\nabla \times (\boldsymbol{M}/r)$ so you can find its Fourier transform from that of $1/r$.]

Solution: Writing the amplitude, $\mathcal{M} = -ie\tilde{\boldsymbol{A}}(\boldsymbol{q}) \cdot \bar{u}'\boldsymbol{\gamma}u$, where $q = p' - p$. So we need to Fourier transform the vector potential

$$\tilde{\boldsymbol{A}} = \int d^3x e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \frac{\nabla \times \boldsymbol{M}}{r} = i\boldsymbol{q} \times \boldsymbol{M} \int d^3x \frac{e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}}{r} = 4\pi i \frac{\boldsymbol{q} \times \boldsymbol{M}}{\boldsymbol{q}^2}$$

Then

$$\mathcal{M} = 4\pi e \frac{\boldsymbol{q} \times \boldsymbol{M}}{\boldsymbol{q}^2} \cdot \bar{u}'\boldsymbol{\gamma}u = \frac{16\pi e \lambda \delta_{\lambda'\lambda}}{\boldsymbol{q}^2} \chi_\lambda^\dagger(\boldsymbol{p}')\boldsymbol{\sigma} \cdot (\boldsymbol{q} \times \boldsymbol{M})\chi_\lambda(\boldsymbol{p})$$

Because of helicity conservation we may assume $\lambda' = \lambda = \pm 1/2$ and find using the projector trick

$$|\mathcal{M}|^2 = \frac{64\pi^2 e^2 \boldsymbol{p}^2}{\boldsymbol{q}^4} \mathrm{Tr}\left[\boldsymbol{\sigma} \cdot (\boldsymbol{q} \times \boldsymbol{M}) \frac{1 + 2\lambda \hat{\boldsymbol{p}}' \cdot \boldsymbol{\sigma}}{2} \boldsymbol{\sigma} \cdot (\boldsymbol{q} \times \boldsymbol{M}) \frac{1 + 2\lambda \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}}{2}\right] \tag{5}$$

$$= \frac{64\pi^2 e^2 \boldsymbol{p}^2}{\boldsymbol{q}^4} \mathrm{Tr}\Bigg[(\boldsymbol{q} \times \boldsymbol{M})^2 \frac{1 - 2\lambda \hat{\boldsymbol{p}}' \cdot \boldsymbol{\sigma}}{2} \frac{1 + 2\lambda \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}}{2}$$

$$+ 2\lambda \hat{\boldsymbol{p}}' \cdot (\boldsymbol{q} \times \boldsymbol{M}) \boldsymbol{\sigma} \cdot (\boldsymbol{q} \times \boldsymbol{M}) \frac{1 + 2\lambda \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}}{2}\Bigg] \tag{6}$$

$$= \frac{64\pi^2 e^2 \boldsymbol{p}^2}{\boldsymbol{q}^4} \left[(\boldsymbol{q} \times \boldsymbol{M})^2 \frac{1 - \hat{\boldsymbol{p}}' \cdot \hat{\boldsymbol{p}}}{2} + \boldsymbol{p}^2 [(\hat{\boldsymbol{p}} \times \hat{\boldsymbol{p}}') \cdot \boldsymbol{M}]^2\right] \tag{7}$$

For $\boldsymbol{p}, \boldsymbol{p}'$ perpendicular to $\boldsymbol{M}$, $(\boldsymbol{q} \times \boldsymbol{M})^2 = \boldsymbol{q}^2 \boldsymbol{M}^2$ and $[(\hat{\boldsymbol{p}} \times \hat{\boldsymbol{p}}') \cdot \boldsymbol{M}]^2 = \boldsymbol{M}^2 \sin^2\phi$, with ϕ the scattering angle. Also $\boldsymbol{q}^2 = 2\boldsymbol{p}^2(1 - \cos\phi)$, so

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{16\pi^2} = \frac{e^2 \boldsymbol{M}^2}{(1 - \cos\phi)^2} \left[(1 - \cos\phi)^2 + \sin^2\phi\right] = \frac{e^2 \boldsymbol{M}^2}{\sin^2(\phi/2)}$$