Quantum Field Theory

Solution Set 9

Due: 22 March 2019

Reading: Lecture Notes, Chapters 5-6.

32. Our class discussion of the propagator for the Dirac field left out a lot of details. Following the strategies sketched in the notes give a detailed and complete derivation of

a) Equations (5.95) and (5.96).

Solution: We start by recalling that under a Lorentz transformation

$$e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}u_\lambda(\boldsymbol{p}) = \sum_{\lambda'} u_{\lambda'}(\Lambda\boldsymbol{p})D_{\lambda'\lambda} \tag{1}$$

where D is a unitary matrix on spin space. It then follows that

$$e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\sum_\lambda u_\lambda(\boldsymbol{p})\bar{u}_\lambda(\boldsymbol{p})e^{+i\lambda_{\mu\nu}\sigma^{\mu\nu}/4} = \sum_\lambda u_\lambda(\Lambda\boldsymbol{p})\bar{u}_\lambda(\Lambda\boldsymbol{p}) \tag{2}$$

This shows that $\sum_\lambda u_\lambda(\boldsymbol{p})\bar{u}_\lambda(\boldsymbol{p})$ is a scalar formed from p and γ, and hence has the form $AI + B\gamma\cdot p$. $(m+\gamma\cdot p)u = 0$ then determines it to be $A(m-\gamma\cdot p)$.Then $\sum_\lambda \bar{u}\gamma^0 u = 4\omega$ gives $A\mathrm{Tr}\gamma^0 2\omega = 4A\omega = 4\omega$ or $A = 1$ and establishes (5.95). Next we use $v = i\gamma^2 u^*$ and $\bar{v} = u^T i\gamma^2\gamma^0 = -u^T\gamma^0 i\gamma^2 = -i\bar{v}^*\gamma^2$ we calculate

$$\sum_\lambda v_\lambda(\boldsymbol{p})\bar{v}_\lambda(\boldsymbol{p}) = \gamma^2(m-\gamma^*\cdot p)\gamma^2 = -m-\gamma\cdot p, \tag{3}$$

establishing (5.96)

b) Equation (5.99) starting from (5.97)-(5.98).

Solution: The first step is to use the integral representation (5.88) pf the step function in (5.98) to get

$$\begin{aligned} S_F(x_1-x_2) &= \int \frac{d^4p}{(2\pi)^4 2\omega}\left[\frac{-ie^{ip\cdot(x_1-x_2)}(m-\gamma\cdot\hat{p})}{\omega-p^0-i\epsilon} + \frac{-ie^{ip\cdot(x_2-x_1)}(m+\gamma\cdot\hat{p})}{\omega-p^0-i\epsilon}\right] \\ &= \int \frac{d^4p}{(2\pi)^4 2\omega}e^{ip\cdot(x_1-x_2)}\left[\frac{-i(m-\gamma\cdot\hat{p})}{\omega-p^0-i\epsilon} + \frac{-i(m-\boldsymbol{\gamma}\cdot\hat{\boldsymbol{p}}-\omega\gamma^0)}{\omega+p^0-i\epsilon}\right] \end{aligned} \tag{4}$$

In this formula $\hat{p}$ has the same spatial components as p, but $\hat{p}^0 = \omega \neq p^0$. In the last line we changed variables $p \to -p$ in the second term. The last step is to combine the terms in square brackets: For the m and spatial p terms the numerator is multiplied by the sum of the two denominators, 2ω, which cancels the 2ω in teh denominator.

The $\omega\gamma^0$ terms have opposite signs and so get multiplied by the difference of the two denominators, $2p^0$, giving $2\omega p^0$. The 2ω then cancels tht in the denominator, so the coefficient of γ^0 is now p^0. This fives the final result

$$S_F(x_1 - x_2) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x_1-x_2)} \frac{-i(m - \gamma \cdot p)}{m^2 + p^2 - i\epsilon} \tag{5}$$

33. In this exercise we study the procedure for obtaining scattering amplitudes from time ordered products for a Dirac field interacting with an external electromagnetic field $A_e^\mu(x)$.

a) Evaluate the amplitude through first order in A_e

$$\langle out|T[\psi_a(x)\bar{\psi}_b(y)|in\rangle \tag{6}$$

in terms of the free Dirac propagator $S_F = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle$. Notice that the first order correction contains a "disconnected" piece which vanishes because $\langle 0|j^\mu(x)|0\rangle = 0$.

Solution:

$$\begin{aligned}
&\langle out|T[\psi_a(x)\bar{\psi}_b(y)|in\rangle \\
&= \langle 0|T[e^{i\int d^4xQA_\mu(\bar{\psi}_I\gamma^\mu\psi_I}\psi_{Ia}(x)\bar{\psi}_{Ib}(y)]|0\rangle \\
&= S_F(x-y) + iQ\int d^4zA_\mu(z)\langle 0|T[\bar{\psi}_I(z)\gamma^\mu\psi_I(z)\psi_{Ia}(x)\bar{\psi}_{Ib}(y)]|0\rangle \\
&= S_F(x-y)_{ab}(1 + iQ\int d^4zA_\mu(z)\langle 0|j^\mu(z)|0\rangle) \\
&\quad +iQ\int d^4zA_\mu(z)[S_F(x-z)\gamma^\mu S_F(z-y)]_{ab}
\end{aligned} \tag{7}$$

The disconnected term which should vanish is the second term in parentheses on the next to last line, and we drop it in the following.

b) Now carry out the Fourier transform

$$\int d^4xd^4ye^{-iq\cdot x+ip\cdot y}\langle out|T[\psi_a(x)\bar{\psi}_b(y)]|in\rangle \tag{8}$$

of your result with respect ro both x and y. You should obtain

$$\frac{-i(m-p\cdot\gamma)}{m^2+p^2}(2\pi)^4\delta(q-p) + \frac{-i(m-q\cdot\gamma)}{m^2+q^2}iQ\gamma\cdot\tilde{A}(q-p)\frac{-i(m-p\cdot\gamma)}{m^2+p^2} \tag{9}$$

where $\tilde{A}(q) = \int d^4xe^{-iq\cdot x}A(x)$. Scattering information is contained in the second term which displays poles in p^2 and q^2 at $-m^2$. That is when p and q are momenta for a particle of mass m. These poles can have positive or negative energy $p^0 = \pm\omega(\boldsymbol{p})$ or $q^0 = \pm\omega(\boldsymbol{q})$. According to F.T. definitions, $q^0 > 0$ ($q^0 < 0$)describes an outgoing (incoming) particle. but $p^0 > 0$ ($p^0 < 0$) corresponds to an incoming (outgoing) particle.

Solution: We use

$$\int d^4x e^{-iq\cdot x} S_F(x-z) = \frac{-i(m-q\cdot\gamma)}{m^2+q^2} e^{-iz\cdot q}, \qquad \int d^4y e^{ip\cdot y} S_F z - y) = \frac{-i(m-p\cdot\gamma)}{m^2+p^2} e^{iz\cdot p}$$

Putting $z = y$ in the first of these we compute

$$\int d^4x d^4y e^{-iq\cdot x + ip\cdot y} S_F(x-y) = \frac{-i(m-p\cdot\gamma)}{m^2+p^2}(2\pi)^4\delta(q-p) \tag{10}$$

which is the first desired term. Applying the double Fourier transform to the last term gives

$$\frac{-i(m-q\cdot\gamma)}{m^2+q^2} iQ\gamma\cdot \int d^4z A(z) e^{-iz\cdot(q-p)} \frac{-i(m-p\cdot\gamma)}{m^2+p^2} \tag{11}$$

which is the desired second term with $\tilde{A}(q-p) = \int d^4z A(z) e^{-iz\cdot(q-p)}$

c) Referring to the results proved in problem 32 above, depending on the energy sign choices, show that the residue of the poles involves the amplitude for one of the processes

$$\begin{aligned} \text{Particle Scattering}: &\qquad \bar{u}(\boldsymbol{q}) iQ\gamma\cdot A u(\boldsymbol{p}) \\ \text{Antiparticle Scattering}: &\qquad \bar{v}(-\boldsymbol{q}) iQ\gamma\cdot A v(-\boldsymbol{p}) \\ \text{Pair Production}: &\qquad \bar{u}(\boldsymbol{q}) iQ\gamma\cdot A v(-\boldsymbol{p}) \\ \text{Pair Annihilation}: &\qquad \bar{v}(-\boldsymbol{q}) iQ\gamma\cdot A u(\boldsymbol{p}) \end{aligned}$$

where $u, \bar{v}$ describe incoming and $\bar{u}, v$ describe outgoing particles.

Solution: According to problem 32 if $p^0 = +\omega(\boldsymbol{p})$, then $m - \gamma\cdot p = \sum_\lambda u_\lambda(\boldsymbol{p})\bar{u}_\lambda(\boldsymbol{p})$. Because of the way p is defined, positive p^0 describes an incoming particle. Factorizing the residue of the pole on helicity sates shows that the contribution of the p leg of the diagram to scattering replaces the propagator with the spinor u which would be supplied by the state $b_{in}^\dagger|in\rangle$ corresponding to an incoming particle.. If $p^0 = -\omega$ on the other hand the identity is $m - \gamma\cdot p = -\sum_\lambda v_\lambda(-\boldsymbol{p})\bar{v}_\lambda(\boldsymbol{p})$. With $p^0 < 0$ the leg is an outgoing antiparticle, since the propagator is replaced by the charge conjugate spinor v which would be supplied by the state $\langle out|d_{out}$. Applying these same considerations to the q leg we see that the propagator is replaced by the spinor $\bar{u}(\boldsymbol{q})$ and describes an outgoing particle when $q^0 = +\omega$ i.e. the state $\langle out|_{out}$. But when $q^0 = -\omega$ it is replaced by $\bar{v}(-\boldsymbol{q})$ and describes an incoming antiparticle, by the state $b_{in}^\dagger|in\rangle$. Then of the four listed processes the first has $q^0, p^0 > 0$, the second has $q^0, p^0 < 0$, the third has $q^0 > 0, p^0 < 0$, and the fourth has $q^0 < 0, p^0 > 0$.

34. Prove, without using perturbation theory, that the propagator for the Dirac field ψ in the presence of an external electromagnetic field A_μ,

$$S_A(x,y) \equiv \frac{\langle out|T[\psi_a(x)\bar{\psi}_b(y)]|in\rangle}{\langle out|in\rangle},$$

is a Green's function for the differential operator $[i\gamma\cdot\partial + q\gamma\cdot A - m]$:

$$(-i\gamma\cdot(\partial - iqA) + m)_{ac}\, S_{A\ cb}(x,y) = -i\delta_{ab}\delta(x-y) \tag{12}$$

You will need the facts that the operator $\psi(x)$ obeys the Dirac equation in the external field $A(x)$, that $d\theta(t)/dt = \delta(t)$ and that the anticommutation relations for $\psi(x), \bar{\psi}(y)$ at $x^0 = y^0$ are not changed by the presence of A. This result gives an alternative starting point for the weak field expansion. Confirm that the first order correction agrees with that you obtained in problem 33a.

Solution:

$$\frac{\langle out|T[\psi_\alpha(x)\bar{\psi}_\beta(y)]|in\rangle}{\langle out|in\rangle} \equiv \theta(x^0-y^0)\frac{\langle out|\psi_\alpha(x)\bar{\psi}_\beta(y)|in\rangle}{\langle out|in\rangle} - \theta(y^0-x^0)\frac{\langle out|\bar{\psi}_\beta(y)\psi_\alpha(x)|in\rangle}{\langle out|in\rangle}$$

Since $[i\gamma\cdot\partial + q\gamma\cdot A - m]\psi = 0$, applying this operator to both sides of equation gives on the right only contributions of the time derivative on the step functions:

$$[i\gamma\cdot\partial + q\gamma\cdot A - m]_{\gamma\alpha}\frac{\langle out|T[\psi_\alpha(x)\bar{\psi}_\beta(y)]|in\rangle}{\langle out|in\rangle} \tag{13}$$

$$= \quad i\gamma^0_{\gamma\alpha}\delta(x^0-y^0)\frac{\langle out|(\psi_\alpha(x)\bar{\psi}_\beta(y) + \bar{\psi}_\beta(y)\psi_\alpha(x))|in\rangle}{\langle out|in\rangle} \tag{14}$$

$$= \quad i\gamma^0_{\gamma\alpha}\delta(x^0-y^0)\frac{\langle out|\gamma^0_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y})|in\rangle}{\langle out|in\rangle} \tag{15}$$

$$= \quad i\delta_{\gamma\beta}\delta(x-y) \tag{16}$$

which is what was to be shown.

35. e^+e^- **Pair production in an external EM field**

a) Write down the lowest order contribution to the amplitude for producing an electron-positron pair in the presence of an external field A_μ. Which Fourier components of $A_\mu(x)$ contribute to pair production to lowest order in qA? What will happen to the vacuum persistence amplitude $\langle out|in\rangle$ if pair production is possible?

Solution:

$$\mathcal{M} = \langle 0|d_{\lambda_1}(\vec{q}_1)b_{\lambda_2}(\vec{q}_2)iQ\int d^4x A\cdot\bar{\psi}\gamma\psi|0\rangle = -ie\bar{u}_{\lambda_2}(\vec{q}_2)\gamma\cdot\tilde{A}(q_1+q_2)v_{\lambda_1}(\vec{q}_1).$$

There can be pair production provided there are Fourier components $q = q_1 + q_2$ with $q^2 \leq -4m^2$, $q^0 > 0$. These are the conditions for $q = q_1 + q_2$ to be the four momentum of a particle antiparticle pair. Note that since the frequency $q^0 = \omega_1 + \omega_2 > 2m$ the conditions for adiabatic change are badly violated. If the probability of pair production is nonzero, the probability of vacuum persistence $|\langle out|in\rangle|^2 < 1$ (This could *not* happen for adiabatic change!)

b) Calculate the differential probability of producing a pair with momenta $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and unobserved helicities, in terms of $\tilde{A}_\mu$, the Fourier transform of $A_\mu(x)$. [You will need

$$\sum_\lambda u_\lambda(p)\overline{u}_\lambda(p) = m - \gamma \cdot p, \qquad \sum_\lambda v_\lambda(p)\overline{v}_\lambda(p) = -m - \gamma \cdot p$$

and the formulae for the trace of products of γ matrices. Remember that $\tilde{A}$ is complex and that $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$.] For the special case $\boldsymbol{q}_1 = \boldsymbol{q}$, $\boldsymbol{q}_2 = -\boldsymbol{q}$, $q_1^0 = q_2^0 = \omega(\boldsymbol{q})$, verify that your answer is positive and that it does not depend on $\tilde{A}^0$, and express it in terms of Fourier components of the electromagnetic field $F_{\mu\nu}$. [This verifies the gauge invariance of the calculation.]

:**Solution**

$$dP = \sum_{\lambda_1,\lambda_2} \frac{d^3q_1 d^3q_2}{(2\pi)^6 4\omega_1\omega_2}|\mathcal{M}|^2 = e^2\frac{d^3q_1 d^3q_2}{(2\pi)^6 4\omega_1\omega_2}\mathrm{Tr}\gamma\cdot\tilde{A}(-m-\gamma\cdot q_1)\gamma\cdot\tilde{A}^*(m-\gamma\cdot q_2) \tag{17}$$

$$= e^2\frac{d^3q_1 d^3q_2}{(2\pi)^6\omega_1\omega_2}\left(-\frac{(q_1+q_2)^2\tilde{A}\cdot\tilde{A}^*}{2} + q_1\cdot\tilde{A}q_2\cdot\tilde{A}^* + q_1\cdot\tilde{A}^*q_2\cdot\tilde{A}\right) \tag{18}$$

For $\vec{q}_1 = -\vec{q}_2 = \vec{q}$, $q_1^0 = q_2^0 = \omega$

$$dP = e^2\frac{d^3q_1 d^3q_2}{(2\pi)^6\omega^2}\left(2\omega^2\tilde{A}\cdot\tilde{A}^* - 2\vec{q}\cdot\vec{\tilde{A}}\vec{q}\cdot\vec{\tilde{A}}^* + 2\omega^2\tilde{A}^0\tilde{A}^{0*}\right) \tag{19}$$

$$= 2e^2\frac{d^3q_1 d^3q_2}{(2\pi)^6}\left(\vec{\tilde{A}}\cdot\vec{\tilde{A}}^* - \frac{\vec{q}}{\omega}\cdot\vec{\tilde{A}}\frac{\vec{q}}{\omega}\cdot\vec{\tilde{A}}^*\right) \tag{20}$$

this is > 0 because $|\vec{q}| < \omega$. The electric field $\vec{E} = -\nabla A^0 - \dot{\vec{A}} \to -i(\mathbf{q}_1+\mathbf{q}_2)\tilde{A}^0 + 2i\omega\vec{\tilde{A}} = 2i\omega\vec{\tilde{A}}$ so $\vec{\tilde{A}} = \vec{\tilde{E}}/2i\omega$. Thus

$$dP = e^2\frac{d^3q_1 d^3q_2}{2(2\pi)^6\omega^2}\left(\vec{\tilde{E}}\cdot\vec{\tilde{E}}^* - \frac{\vec{q}}{\omega}\cdot\vec{\tilde{E}}\frac{\vec{q}}{\omega}\cdot\vec{\tilde{E}}^*\right) \tag{21}$$