Quantum Field Theory

Solution Set 10

Due: 29 March 2019

Reading: Lecture Notes, Chapters 6-7.

36 and 37. We shall soon find, when we complete the quantization of the electromagnetic field, that the photon propagator can be taken to be

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{-i\eta_{\mu\nu}e^{i(x-y)\cdot p}}{p^2-i\epsilon}.$$

Anticipatiing this result, the Feynman amplitudes for elastic e^+e^- and e^-e^- scattering are

$$\begin{aligned}\mathcal{M}_{e^+e^-} &= -ie^2\left[\frac{\bar v(q)\gamma_\mu v(q')\bar u(p')\gamma^\mu u(p)}{t} - \frac{\bar u(p')\gamma_\mu v(q')\bar v(q)\gamma^\mu u(p)}{s}\right]\\ \mathcal{M}_{e^-e^-} &= ie^2\left[\frac{\bar u(q')\gamma_\mu u(q)\bar u(p')\gamma^\mu u(p)}{t} - \frac{\bar u(p')\gamma_\mu u(q)\bar v(q')\gamma^\mu u(p)}{u}\right] \qquad (1)\end{aligned}$$

where $s=-(q+p)^2$, $t=-(q'-q)^2$ and $u=-(p'-q)^2$. Draw the Feynman diagrams depicting the two terms on the right side. Then calculate, to lowest order, the differential cross-section for

36. e^+e^- scattering (Bhabha scattering) and

Solution: The amplitude for the two diagrams for e^+e^- scattering is

$$\mathcal{M}_{\rm Bhabba} = -ie^2\left[\frac{\bar v\gamma_\mu v'\bar u'\gamma^\mu u}{t} - \frac{\bar u'\gamma_\mu v'\bar v\gamma^\mu u}{s}\right]$$

where the primed spinors refer to final state particles and unprimed to initial state particles. Let p, p' be electron momenta and q, q' positron momenta, so $t=-(q'-q)^2$ and $s=-(q+p)^2$. Summing over all particle spins converts $|\mathcal{M}|^2$ to traces:

$$\begin{aligned}\Sigma|\mathcal{M}|^2 = e^4\Big[&\frac{1}{t^2}\mathrm{Tr}\{\gamma_\mu(m+q'\cdot\gamma)\gamma_\lambda(m+q\cdot\gamma)\}\mathrm{Tr}\{\gamma^\mu(m-p\cdot\gamma)\gamma^\lambda(m-p'\cdot\gamma)\} \qquad (2)\\ &+\frac{1}{s^2}\mathrm{Tr}\{\gamma_\mu(m+q'\cdot\gamma)\gamma_\lambda(m-p'\cdot\gamma)\}\mathrm{Tr}\{\gamma^\mu(m-p\cdot\gamma)\gamma^\lambda m+q\cdot\gamma)\} \qquad (3)\\ &-\frac{1}{st}\mathrm{Tr}\{\gamma_\mu(m+q'\cdot\gamma)\gamma_\lambda(m+q\cdot\gamma)\gamma^\mu(m-p\cdot\gamma)\gamma^\lambda(m-p'\cdot\gamma)\}\\ &-\frac{1}{st}\mathrm{Tr}\{\gamma_\mu(m+q'\cdot\gamma)\gamma_\lambda(m-p'\cdot\gamma)\gamma^\mu(m-p\cdot\gamma)\gamma^\lambda(m+q\cdot\gamma)\}\Big] \qquad (4)\end{aligned}$$

We list some intermediate trace results:

$$\begin{aligned}
&\mathrm{Tr}\{\gamma_\mu q'\cdot\gamma\gamma_\lambda q\cdot\gamma\}\mathrm{Tr}\{\gamma^\mu p\cdot\gamma\gamma^\lambda p'\cdot\gamma\} = 16(q'_\mu q_\lambda + q'_\lambda q_\mu - \eta_{\mu\lambda} q\cdot q')(p'_\mu p_\lambda + p'_\lambda p_\mu - \eta_{\mu\lambda} p\cdot p') \\
&= 32(q'\cdot p' q\cdot p + q'\cdot p q\cdot p') \\
&= 8((s-2m^2)^2 + (s+t-2m^2)^2) = 16s^2 + 16st + 8t^2 - 32m^2(2s+t) + 64m^4 \qquad (5) \\
&\mathrm{Tr}\{\gamma_\mu q'\cdot\gamma\gamma_\lambda p'\cdot\gamma\}\mathrm{Tr}\{\gamma^\mu p\cdot\gamma\gamma^\lambda q\cdot\gamma\} = 16t^2 + 16st + 8s^2 - 32m^2(2t+s) + 64m^4 \\
&\mathrm{Tr}\{\gamma_\mu q'\cdot\gamma\gamma_\lambda q\cdot\gamma\gamma^\mu p\cdot\gamma\gamma^\lambda p'\cdot\gamma\} = \mathrm{Tr}\{\gamma_\mu q'\cdot\gamma\gamma_\lambda p'\cdot\gamma\gamma^\mu p\cdot\gamma\gamma^\lambda q\cdot\gamma\} \qquad (6) \\
&= 2\mathrm{Tr}\{q\cdot\gamma\gamma_\lambda q'\cdot\gamma p\cdot\gamma\gamma^\lambda p'\cdot\gamma\} = 8q'\cdot p\mathrm{Tr}\{q\cdot\gamma p'\cdot\gamma\} \\
&= -32q'\cdot pq\cdot p' = -8(s+t-2m^2)^2 \qquad (7)
\end{aligned}$$

Putting everything together and simplifying,

$$\Sigma|\mathcal{M}|^2 = 16e^4\left[\frac{s^2}{t^2} + \frac{t^2}{s^2} + 2\frac{s}{t} + 2\frac{t}{s} + 3 - 4m^2\left(\frac{s}{t^2} + \frac{t}{s^2}\right) + 4\left(\frac{m^4}{t^2} + \frac{m^4}{s^2} - \frac{m^4}{st}\right)\right]$$

Averaging over initial spins means to divide this sum by 4. Then plugging into the cross section formula we get

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{e^4}{16\pi^2 s}\left[\frac{s^2}{t^2} + \frac{t^2}{s^2} + 2\frac{s}{t} + 2\frac{t}{s} + 3 - 4m^2\left(\frac{s}{t^2} + \frac{t}{s^2}\right) + 4\left(\frac{m^4}{t^2} + \frac{m^4}{s^2} - \frac{m^4}{st}\right)\right] \\
&= \frac{\alpha^2}{s}\left[\frac{s^2}{t^2} + \frac{t^2}{s^2} + 2\frac{s}{t} + 2\frac{t}{s} + 3 - 4m^2\left(\frac{s}{t^2} + \frac{t}{s^2}\right) + 4\left(\frac{m^4}{t^2} + \frac{m^4}{s^2} - \frac{m^4}{st}\right)\right] \qquad (8)
\end{aligned}$$

$t = -(p-p')^2 = -2\boldsymbol{p}^2(1-\cos\theta) = -(s-4m^2)\sin^2(\theta/2)$. Thus at low energy $-t \ll s$ and

$$\frac{d\sigma}{d\Omega} \sim \frac{\alpha^2 m^2}{16|\boldsymbol{p}|^4\sin^4(\theta/2)} \qquad (9)$$

which is just the non-relativistic Rutherford scattering result. On the other hand at high energy we can neglect the electron mass and obtain

$$\frac{d\sigma}{d\Omega} \sim \frac{\alpha^2}{s}\left[\frac{1}{\sin^4(\theta/2)} + \sin^4(\theta/2) - \frac{2}{\sin^2(\theta/2)} - 2\sin^2(\theta/2) + 3\right]$$

The $\theta \to 0$ divergence comes from the $1/t$ in the photon exchange diagram.

37. e^-e^- scattering (Moller scattering)

Solution: For e^-e^- scattering, we see that the spin summed square of the amplitude can be obtrained from that for e^+e^- by substituting $s \to u$:

$$\Sigma|\mathcal{M}|^2 = 16e^4\left[\frac{u^2}{t^2} + \frac{t^2}{u^2} + 2\frac{u}{t} + 2\frac{t}{u} + 3 - 4m^2\left(\frac{u}{t^2} + \frac{t}{u^2}\right) + 4\left(\frac{m^4}{t^2} + \frac{m^4}{u^2} - \frac{m^4}{ut}\right)\right]$$

so the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s}\left[\frac{u^2}{t^2} + \frac{t^2}{u^2} + 2\frac{u}{t} + 2\frac{t}{u} + 3 - 4m^2\left(\frac{u}{t^2} + \frac{t}{u^2}\right) + 4\left(\frac{m^4}{t^2} + \frac{m^4}{u^2} - \frac{m^4}{ut}\right)\right] \qquad (10)$$

Now $u = -(s - 4m^2)\cos^2(\theta/2)$, giving $u/t = \cot^2(\theta/2)$. At low energies $-u, -t \ll s \approx 4m^2$ so at fixed angle the last term dominates

$$\frac{d\sigma}{d\Omega} \sim \frac{\alpha^2 m^2}{16|\boldsymbol{p}|^4}\frac{1 - 3\sin^2(\theta/2)\cos^2(\theta/2)}{\sin^4(\theta/2)\cos^4(\theta/2)} = \frac{\alpha^2 m^2}{4|\boldsymbol{p}|^4}\frac{1 + 3\cos^2\theta}{\sin^4\theta}$$

At high energy we can neglect the electron mass obtaining

$$\frac{d\sigma}{d\Omega} \sim \frac{\alpha^2}{s}\left[\frac{u^2}{t^2} + \frac{t^2}{u^2} + 2\frac{u}{t} + 2\frac{t}{u} + 3\right] \tag{11}$$

$$\sim \frac{\alpha^2}{s}\left[\cot^4(\theta/2) + \tan^4(\theta/2) + 2\cot^2(\theta/2) + 2\tan^2(\theta/2) + 3\right] \tag{12}$$

$$\sim \frac{\alpha^2}{s}\left[\frac{1}{\sin^4(\theta/2)} + \frac{1}{\cos^4(\theta/2)} + 1\right] \tag{13}$$

for the case that both particles in the initial state are unpolarized and the final spins are unobserved. Notice that the expressions for the squares of the amplitudes summed over spins for these two processes are simply related to each other by the substiturions $q \leftrightarrow -q'$. and interchanging some of the 4-momenta. Thus the traces need to be calculated for only one of the two processes. Give the differential cross section for both processes in the Center of Mass frame. Discuss the limiting cases of low and high energy for both processes.

38. S, Problem 48.5

Solution:

a) The vertex for this decay is $ic_1 G_F f_\pi (p_1 + p_2)_\mu \gamma^\mu (1 - \gamma_5)$. and the amplitude is therefore

$$\mathcal{M} = ic_1 G_F f_\pi \bar{u}_1 i(p_1 + p_2)\cdot\gamma(1 - \gamma_5)v_2 = m_\mu c_1 G_F f_\pi \bar{u}_1(1 - \gamma_5)v_2 \tag{14}$$

$$|\mathcal{M}|^2 = c_1^2 G_F^2 f_\pi^2 m_\mu^2 \mathrm{Tr}(m - \gamma\cdot p_1)(1 - \gamma_5)(-\gamma\cdot p_2)(1 + \gamma_5)$$

$$= 2c_1^2 G_F^2 f_\pi^2 m_\mu^2 \mathrm{Tr}(m - \gamma\cdot p_1)(-\gamma\cdot p_2)(1 + \gamma_5) = -8c_1^2 G_F^2 f_\pi^2 m_\mu^2 p_1\cdot p_2 \tag{15}$$

Now $-2p_1\cdot p_2 = -(p_1 + p_2)^2 - m_\mu^2 = m_\pi^2 - m_\mu^2$, so

$$\mathcal{M} = 4c_1^2 G_F^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2) \tag{16}$$

$$\Gamma = \int \frac{p_2 dp_2}{(8\pi)\omega_1 m_\pi}\delta(\sqrt{m_\mu^2 + p_2^2} + p_2 - m_\pi) 4c_1^2 G_F^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)$$

$$= \frac{p_2}{(2\pi)m_\pi}(\omega_1 + p_2)^{-1} c_1^2 G_F^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)$$

$$= \frac{1}{4\pi m_\pi^3} c_1^2 G_F^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)^2 \tag{17}$$

b) Putting in the numbers $f_\pi \approx 93.1$ MeV.

39. **Feynman rules for a Majorana field.** Recall that a Majorana field has the expansion and anticommutation relations

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega}} \sum_\lambda \left(b_\lambda(\boldsymbol{p}) u_\lambda(\boldsymbol{p}) e^{ix\cdot p} + b^\dagger_\lambda(\boldsymbol{p}) v_\lambda(\boldsymbol{p}) e^{-ip\cdot x} \right) \tag{18}$$

$$\{\psi^a(\boldsymbol{x}), \psi^b(\boldsymbol{x}')\} = (i\gamma^2)_{ab}\delta(\boldsymbol{x}-\boldsymbol{x}'), \qquad \psi^\dagger = \psi^T i\gamma^2 \tag{19}$$

a) Show that written as $\langle 0|T[\psi_a(x)\bar{\psi}_b(y)]|0\rangle$ the propagator is the same as the Dirac propagator $S_F(x-y)_{ab}$. Obtain from this result an expression for $\langle 0|T[\psi_a(x)\psi_b(y)]|0\rangle$, which is nonzero for a Majorana field (but not for a Dirac field!).

Solution: We write out

$$\begin{aligned}\langle 0|T[\psi_a(x)\bar{\psi}_b(0)]|0\rangle &= \int \frac{d^3p d^3q}{(2\pi)^3\sqrt{4\omega_p\omega_q}} (\theta(t)\langle 0|b_\lambda(\boldsymbol{p})u^a_\lambda(\boldsymbol{p})e^{ix\cdot p} b^\dagger_\lambda(\boldsymbol{q})\bar{u}^b_\lambda(\boldsymbol{q})|0\rangle \\ &\quad -\theta(-t)\langle 0|b_\lambda(\boldsymbol{q})\bar{v}^b_\lambda(\boldsymbol{q})e^{-ix\cdot p} b^\dagger_\lambda(\boldsymbol{p}) v^a_\lambda(\boldsymbol{p})|0\rangle) \\ &= \int \frac{d^3p}{(2\pi)^3 2\omega_p} (\theta(t)(u_\lambda\bar{u}_\lambda)_{ab} e^{ix\cdot p} - \theta(-t)(v_\lambda\bar{v}_\lambda)_{ab}e^{-ix\cdot p})\end{aligned} \tag{20}$$

The last line is the same combination of terms that make up the Dirac propagator, so they are identical. But for the Majorana field $\bar{\psi} = \psi^T i\gamma^2\gamma^0$, so

$$\begin{aligned}\langle 0|T[\psi_a(x)\psi_b(y)]|0\rangle &= \langle 0|T[\psi_a(x)\psi^T_b(y)]|0\rangle = \langle 0|T[\psi_a(x)(\bar{\psi}\gamma^0 i\gamma^2)_b(y)]|0\rangle \\ &= (S_F(x-y)\gamma^0 i\gamma^2)_{ab}\end{aligned} \tag{21}$$

b) Consider an interaction with a scalar field ϕ due to a term $(g/2)\phi\bar{\psi}\psi$ in the Hamiltonian in time dependent perturbation theory, and show that, if ϕ is an external field, to lowest order in g the Feynman amplitude $\mathcal{M}$ is $-ig\bar{u}(\boldsymbol{p}')u(\boldsymbol{p})\tilde{\phi}(p'-p)$ Note that this would be the amplitude for a Dirac interaction term $g\phi\bar{\psi}\psi$. Explain the factor of 2 difference in the Dirac and Majorana interaction terms.

Solution: Here I put $p' = q$ for ease of writing. To calculate the scattering amplitude to lowest order in pertubation theory we need

$$\begin{aligned}\{b_\lambda, \psi(x)\}(2\pi)^{3/2}\sqrt{2\omega} &= v_\lambda e^{-ix\cdot p} \\ \{b_\lambda, \bar{\psi}(x)\}(2\pi)^{3/2}\sqrt{2\omega} &= \bar{u}_\lambda e^{-ix\cdot p} \\ \{\psi(x), b^\dagger_\lambda\}(2\pi)^{3/2}\sqrt{2\omega} &= u_\lambda e^{ix\cdot p} \\ \{\bar{\psi}(x), b^\dagger_\lambda\}(2\pi)^{3/2}\sqrt{2\omega} &= \bar{v}_\lambda e^{ix\cdot p}\end{aligned} \tag{22}$$

Then the Feynman amplitude is

$$\mathcal{M} = -i\frac{g}{2}\int d^4x e^{ix\cdot(p-q)} \left(\bar{u}(q)u(p) - \bar{v}(p)v(q)\right)\phi(x) = -i\frac{g}{2}\left(\bar{u}(q)u(p) - \bar{v}(p)v(q)\right)\tilde{\phi}(q-p) \tag{23}$$

but we can express

$$\bar{v}(p)v(q) = u^T(p)i\gamma^2\gamma^0 i\gamma^2 u^*(q) = -u^T(p)\gamma^0 u^*(q) = -u^\dagger(q)\gamma^{0T}u(p) = -\bar{u}u \quad (24)$$

So the two terms are equal and so

$$\mathcal{M} = -2i\frac{g}{2}\bar{u}(q)u(p)\tilde{\phi}(q-p) = -ig\bar{u}(q)u(p)\tilde{\phi}(q-p) \quad (25)$$

The factor of 1/2 in the Majorana interaction term compensates for the fact that there are twice as many nonzero contractions as for the Dirac interaction term.

c) Now assume ϕ is a quantum field, and write down the expression for the (connected) second order correction to the momentum space ϕ propagator $\int d^4x e^{iq\cdot x}\langle 0|T\phi(x)\phi(0)|0\rangle$. Compare the expressions for the Majorana and Dirac field and explain the differences. Do all of the position space integrals, but leave the last momentum integral undone.

Solution: The second oreder correction is given by

$$\begin{aligned}
&\frac{1}{2}\left(\frac{-ig}{2}\right)^2 \int d^4x d^4y \langle 0|T\phi(z)\phi(0)\phi(x)\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y)|0\rangle \\
&= \frac{-g^2}{4}\int d^4x d^4y \Delta_F(z-x)\Delta_F(-y)\langle 0|T\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y)|0\rangle \\
&= \frac{-g^2}{4}\int d^4x d^4y \Delta_F(z-x)\Delta_F(-y)\left(-2\mathrm{Tr}S_F(x-y)S_F(y-x)\right) \\
&= \frac{g^2}{2}\int d^4x d^4y \Delta_F(z-x)\Delta_F(-y)\mathrm{Tr}S_F(x-y)S_F(y-x) \quad (26)
\end{aligned}$$

where a factor of 2 has been cancelled in the second line by the two ways of contracting the scalar fields.To get the third line we note the two ways to contract a pair of ψ's from the product

$$\begin{aligned}
\psi_a(x)\bar{\psi}_b(y)\psi_b(y) &\to S_F(x-y)_{ab}\psi_b(y) - \bar{\psi}_b(y)(S_F(x-y)\gamma^0 i\gamma^2)_{ab} \\
-\bar{\psi}_b(y)(S_F(x-y)i\gamma^2)_{ab} &= -(\gamma^0 i\gamma^2\psi(y))_b(y)(S_F(x-y)\gamma^0 i\gamma^2)_{ab} \\
&= -(S_F(x-y)\gamma^0 i\gamma^2)_{ab}(\gamma^0 i\gamma_2\psi)_b(y) \\
&= -(S_F(x-y)\gamma^0 i\gamma^2\gamma^0 i\gamma_2)_{ab}\psi_b(y) \\
&= S_F(x-y)_{ab}\psi_b(y) \quad (27)
\end{aligned}$$

Fourier transforming our result

$$
\begin{aligned}
&\int d^4 z e^{iq\cdot z} \frac{g^2}{2} \int d^4 x d^4 y \Delta_F(z-x)\Delta_F(-y)\mathrm{Tr} S_F(x-y) S_F(y-x) \\
= \ & \frac{g^2}{2}\frac{-i}{q^2+\mu^2} \int d^4 x d^4 y e^{iq\cdot x} \Delta_F(-y)\mathrm{Tr} S_F(x-y) S_F(y-x) \\
= \ & \frac{g^2}{2}\frac{-i}{q^2+\mu^2} \int d^4 x d^4 y e^{iq\cdot(x+y)} \Delta_F(-y)\mathrm{Tr} S_F(x) S_F(-x) \\
= \ & \frac{g^2}{2}\frac{-i}{q^2+\mu^2}\frac{-i}{q^2+\mu^2} \int d^4 x e^{iq\cdot x} \mathrm{Tr} S_F(x) S_F(-x) \\
= \ & \frac{g^2}{2}\frac{-i}{q^2+\mu^2}\frac{-i}{q^2+\mu^2} \int \frac{d^4 p}{(2\pi)^4} \frac{(-i)^2 \mathrm{Tr}(m-\gamma\cdot p)(m-\gamma\cdot(p+q))}{(m^2+p^2-i\epsilon)(m^2+(p+q)^2-i\epsilon)}
\end{aligned} \tag{28}
$$

This result is half of the corresponding correction with a Dirac fermion. For the latter there would be no 2 in the interaction, and only a single way of contracting the fermions. For the Majorana case there was a net factor of 1/4 from the two interactions and 2 ways to contract out the fermion fields, leaving a net 1/2 compared to the Dirac case.