Quantum Field Theory

Problem Set 11

Due: 5 April 2019

Reading: Lecture Notes, Chapters 8 and 10. (Chapter 9 is optional reading on scattering in external fields).

40. S, Problem 11.4

Solution: The tree level process occurs at second order involving two ABC vertices. The internal propagator contracts away one of the fields leaving a pair each of the other fields to contract with the external particles. Thus the nonvanishing processes have $2A$'s AND $2B$'s; $2A$'s and $2C$'s; or $2B$'s and $2C$'s. Of the listed processes the only ones with nonvanishing trees are $AA \to BB$ and $AB \to AB$, and the internal propagator for both is $-i/(m_C^2 + p^2)$. For $AA \to BB$, $p^2 = -t$ or $-u$, whereas for $AB \to AB$ $p^2 = -s$ or $-t$. Thus

$$\mathcal{M}_{AA\to BB} = -i(ig)^2 \left[\frac{1}{m_C^2 - t} + \frac{1}{m_C^2 - u}\right], \qquad \mathcal{M}_{AB\to AB} = -i(ig)^2 \left[\frac{1}{m_C^2 - s} + \frac{1}{m_C^2 - t}\right]$$

All the other listed processes are 0 at tree level. Note that our $\mathcal{M}$ is i times Srednicki's $\mathcal{T}$.

41. We introduced the nonabelian gauge potential A_μ as a matrix valued generalization of the vector potential of QED, and determined its gauge transformation so that $(\partial_\mu - igA_\mu)\psi = D_\mu\psi$ transforms as $D_\mu\psi \to \Omega(x)D_\mu\psi$ under $\psi \to \Omega\psi$, In QED we construct a gauge invariant field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which enters the EM part of the Lagrangian as $-F_{\mu\nu}F^{\mu\nu}/4$.

a) To construct a nonabelian field strength analogous to that in QED, we cannot insist that it be gauge invariant but we can require that $F_{\mu\nu} \to \Omega F_{\mu\nu}\Omega^{-1}$, so that $\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$ is gauge invariant. Show that if we regard D_μ as a matrix valued differential operator, then under a gauge transformation

$$A_\mu \to \Omega A_\mu \Omega^{-1} - \frac{i}{g}(\partial_\mu\Omega)\Omega^{-1} \tag{1}$$

$D_\mu \to \Omega D_\mu \Omega^{-1}$.

Solution: We examine

$$\begin{aligned}
\Omega D_\mu \Omega^{-1} &= \Omega\partial_\mu\Omega^{-1} - ig\Omega A_\mu\Omega^{-1} = \partial_\mu + \Omega(\partial_\mu\Omega^{-1}) - ig\Omega A_\mu\Omega^{-1} \\
&= \partial_\mu - \Omega(\Omega^{-1}\partial_\mu\Omega\Omega^{-1}) - ig\Omega A_\mu\Omega^{-1} = \partial_\mu - \partial_\mu\Omega\Omega^{-1}) - ig\Omega A_\mu\Omega^{-1} \\
&= \partial - ig\left(\frac{-i}{g}\partial_\mu\Omega\Omega^{-1}) + \Omega A_\mu\Omega^{-1}\right) = \partial_\mu - igA_\mu^\Omega
\end{aligned} \tag{2}$$

which is the covariant derivative with the gauge transformed potential, as desired.

b) From part a) it follows that $D_\mu D_\nu \to \Omega D_\mu D_\nu \Omega^{-1}$ so that it transforms as we wish $F_{\mu\nu}$ to transform. However, it is not yet a satisfactory field because it is a differential operator. Show that $[D_\mu, D_\nu] = D_\mu D_\nu - D_\nu D_\mu$ is a matrix valued field and evaluate it as a function of A and ∂A.

Solution: We calculate

$$\begin{aligned}
[D_\mu, D_\nu] &= [\partial_\mu, \partial_\nu] - ig[\partial_\mu, A_\nu] - ig[A_\mu, \partial_\nu] - g^2[A_\mu, A_\nu] \\
&= -ig\partial_\mu A_\nu + ig\partial_\nu A_\mu - g^2[A_\mu, A_\nu] = -ig(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]) \quad (3)
\end{aligned}$$

The differential operators have cancelled in the final expression, which we can call $-igF_{\mu\nu}$ with F the desired field strength.

c) Assume the gauge group is a Lie group generated by matrices t_a satisfying the Lie algebra

$$[t_a, t_b] = if_{abc}t_c \quad (4)$$

where f_{abc} are called the structure constants of the group. Then we can expand the gauge potential $A_\mu(x) = \sum_a t_a A^a_\mu$. Defining $F^a_{\mu\nu}$ by $[D_\mu, D_\nu] = -ig\sum_a t_a F^a_{\mu\nu}$, express $F^a_{\mu\nu}$ in terms of A^a_μ, its derivatives, and the structure constants.

Solution: We calculate

$$\begin{aligned}
\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] &= \sum_a (t_a(\partial_\mu A^a_\nu - \partial \nu A^a_\mu) - ig\sum_{a,b}[t_a, t_b]A^a_\mu A^b_\nu \\
&= \sum_c t_c \left(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu + gf_{abc}A^a_\mu A^b_\nu\right) \quad (5)
\end{aligned}$$

The right side is just $\sum_c t_c F^c_{\mu\nu}$.

42. Show without using perturbation theory that the function

$$\Delta_A(x, y) \equiv \frac{\langle out|T[\phi(x)\phi^\dagger(y)]|in\rangle}{\langle out|in\rangle}$$

for a charged scalar field ϕ in an external electromagnetic field, is a Green's function for the differential operator $m^2 - (\partial - iqA)^2$:

$$[m^2 - (\partial - iqA)^2]\Delta_A(x, y) = -i\delta(x - y).$$

You will need to use the canonical commutation rules for the charged scalar field and the fact that ϕ satisfies the equation $[m^2 - (\partial - iqA)^2]\phi = 0$.

Solution: Because $[m^2 - (\partial - iqA)^2]\phi = 0$, the only non-zero contributions to $[m^2 - (\partial - iqA)^2]\Delta_A(x, y)$ come from time derivatives acting on the θ functions implicit in the time ordering symbol. the first time derivative gives

$$\partial_0 \Delta_A(x,y) = \delta(x^0 - y^0)\frac{\langle out|[\phi(x),\phi^\dagger(y)]|in\rangle}{\langle out|in\rangle} + \frac{\langle out|T[\dot\phi(x)\phi^\dagger(y)]|in\rangle}{\langle out|in\rangle} \tag{6}$$

$$= \frac{\langle out|T[\dot\phi(x)\phi^\dagger(y)]|in\rangle}{\langle out|in\rangle} \tag{7}$$

because ϕ and $\phi^\dagger$ commute at equal times. Then

$$\partial_0^2 \Delta_A(x,y) = \delta(x^0 - y^0)\frac{\langle out|[\dot\phi(x),\phi^\dagger(y)]|in\rangle}{\langle out|in\rangle} + \frac{\langle out|T[\ddot\phi(x)\phi^\dagger(y)]|in\rangle}{\langle out|in\rangle}$$

$$= -i\delta(x-y) + \frac{\langle out|T[\ddot\phi(x)\phi^\dagger(y)]|in\rangle}{\langle out|in\rangle} \tag{8}$$

by the canonical commutation relations $[\phi^\dagger, \pi^\dagger] = [\dot\phi - iqA_0\phi, \phi^\dagger] = [\dot\phi, \phi^\dagger] = -i\delta$. It follows that

$$[m^2 - (\partial - iqA)^2]\Delta_A(x,y) = -i\delta(x-y).$$

43. **Positron scattering**

a) Write down an explicit formula for

$$\frac{\langle out|d^{out}_{\lambda'}(\boldsymbol{q}')d^{in\dagger}_{\lambda}(\boldsymbol{q})|in\rangle}{\langle out|in\rangle}$$

as an expansion in powers of $e\tilde{A}_\mu(p)$, where the states are positron states.

Solution: Passing to interaction picture

$$\frac{\langle out|d^{out}_{\lambda'}(\boldsymbol{q}')d^{in\dagger}_{\lambda}(\boldsymbol{q})|in\rangle}{\langle out|in\rangle} = \frac{\langle 0|d_{\lambda'}(\vec{q}')Te^{ie\int d^4x A_\mu \bar\psi\gamma^\mu\psi}d^\dagger_\lambda(\vec{q})|0\rangle}{\langle 0|Te^{ie\int d^4x A_\mu\bar\psi\gamma^\mu\psi}|0\rangle}$$

$$= \sum_{n=0}^\infty \frac{(ie)^n}{n!}\langle 0|d_{\lambda'}(\vec{q}')\int d^4x_1\cdots d^4x_n \bar\psi(x_1)A(x_1)\cdot\gamma\psi(x_1)\cdots\bar\psi(x_n)A(x_n)\cdot\gamma\psi(x_n)d^\dagger_\lambda(\vec{q})|0\rangle_{connected}$$

$$= -\sum_{n=0}^\infty (ie)^n \int d^4x_1\cdots d^4x_n \bar{v}_\lambda(\vec{q})e^{iq\cdot x_1}A(x_1)\cdot\gamma S_F(x_1-x_2)A(x_2)\cdot\gamma\cdots S_F(x_{n-1}-x_n)A(x_n)\cdot\gamma v_{\lambda'}(\vec{q}')e^{-iq'\cdot x_n}$$

where the disconnected vacuum diagrams have cancelled and there is a net overall $-$ from Fermi statistics. There are $n!$ equivalent contraction sets. We displayed one and cancelled the $1/n!$.

b) The terms in the expansion of part a) are spinor matrix elements involving $v, \bar v$ rather than $u, \bar u$ as electron scattering would. By using the relation of v to u, $v_\lambda(p) = i\gamma^2\gamma^0\overline{u}_\lambda(p)^T$ and $\overline{v}_\lambda(p) = u^T_\lambda(p)i\gamma^2\gamma^0$ and $(i\gamma^2\gamma^0)\gamma^\mu = -\gamma^{\mu T}(i\gamma^2\gamma^0)$, show explicitly that the positron amplitude in a field A_μ is the same as the electron amplitude in the field $-A_\mu$. [Of course this follows immediately from the charge conjugation properties of the theory, but I want you to follow how it happens in the Feynman diagram amplitudes. Take care with the overall sign.] What is the relation between the differential cross sections for positron and electron scattering in the same field A_μ in Born approximation?

Solution': Now use $v' = i\gamma^2\gamma^0\bar{u}'^T$, $\bar{v} = u^T i\gamma^2\gamma^0$ and transpose the whole summand to get

$$\sum_{n=0}^{\infty}(ie)^n \int d^4x_1 \cdots d^4x_n \bar{u}_{\lambda'}(\vec{q}')e^{iq\cdot x_1} i\gamma^0\gamma^2 A(x_n)\cdot\gamma^T S_F^T(x_{n-1}-x_n)\cdots S_F^T(x_1-x_2)A(x_1)\cdot\gamma^T(-i\gamma^0\gamma^2)u_\lambda(\vec{q})e^{-iq'\cdot x_n}$$

where the overall $-$ is now in front of the second $i\gamma^0\gamma^2$. Now use $i\gamma^0\gamma^2\gamma^T(-i\gamma^0\gamma^2) = -\gamma$ to show

$$i\gamma^0\gamma^2 S_F^T(x)(-i\gamma^0\gamma^2) = -i\int \frac{d^4p}{(2\pi)^4} e^{ix\cdot p} \frac{m+\gamma\cdot p}{m^2+p^2-i\epsilon} = S_F(-x)$$

so the expansion becomes

$$\sum_{n=0}^{\infty}(ie)^n \int d^4x_1 \cdots d^4x_n \bar{u}_{\lambda'}(\vec{q}')e^{-iq'\cdot x_n}(-A(x_n)\cdot\gamma)S_F(x_n-x_{n-1})(-A(x_{n-1})\cdot\gamma)\cdots S_F(x_2-x_1)(-A(x_1)\cdot\gamma)u_\lambda(\vec{q})e^{iq\cdot x_1}$$

which is the expansion for electron scattering in the potential $-A_\mu$. In Born approximation the amplitude is linear in A so the difference in sign for electron and positron scattering is irrelevant for the differential cross section, so they have identical scattering.