Quantum Field Theory

Solution Set 11

Due: 12 April 2019

44. Consider the charged scalar field in the presence of an external electromagnetic field. The Hamiltonian is given in problem 8 (of Set 2), and in problem 42 (of Set 11) you showed that

$$\Delta_A(x,y) \equiv \frac{\langle out|T[\phi(x)\phi^\dagger(y)]|in\rangle}{\langle out|in\rangle}$$

is a Green's function for the differential operator $m^2 - (\partial - iqA)^2$:

$$[m^2 - (\partial - iqA)^2]\Delta_A(x,y) = -i\delta(x-y).$$

a) Show that under $A \to A + \delta A$, $\ln\langle out|in\rangle_A$ changes (to first order in δA) by the amount

$$\delta \ln\langle out|in\rangle_A = i\int d^4x \delta A^\mu(x) \frac{\langle out|j_\mu(x)|in\rangle_A}{\langle out|in\rangle_A},$$

where $j_\mu = -iq(\phi^\dagger\partial_\mu\phi - \phi\partial_\mu\phi^\dagger) - 2q^2A_\mu\phi^\dagger\phi$. [Hint: first obtain the analogue, for the charged scalar field, of Eq.(7.20) in the lecture notes.]

Solution: The analogue of Eq.(7.20) for the scalar system reads

$$\delta U(t,t_0) = U(t,t_0)i\int_{t_0}^{t} dt' d^3x' \delta A_\mu(x') U^\dagger(t') j_S^\mu(\boldsymbol{x}') U(t')$$

where

$$\begin{aligned} j_S^k(\boldsymbol{x}') &= -iq(\phi_S^\dagger\partial_k\phi_S - \phi_S\partial_k\phi_S^\dagger) - 2q^2A_k\phi_S^\dagger\phi_S \\ j_{0S}(\boldsymbol{x}') &= -iq(\phi_S^\dagger\pi_S^\dagger - \phi_S\pi_S) \end{aligned}$$

are the Schrödinger picture current operators. Then conjugation by $U(t',t_0)$ converts these to Heisenberg picture operators, for which $\pi = (\partial_0 + iqA_0)\phi^\dagger$ so that

$$U^\dagger(t') j_S^\mu(\boldsymbol{x}') U(t') = -iq(\phi^\dagger\partial_\mu\phi - \phi\partial_\mu\phi^\dagger) - 2q^2A_\mu\phi^\dagger\phi.$$

Then taking outin matrix element of the change equation gives the desired result.

b) Provisionally make the identification

$$\frac{\langle out|j^\mu(x)|in\rangle_A}{\langle out|in\rangle_A} = \lim_{y\to x}\left\{-iq(\partial_x^\mu - iqA(x))\frac{\langle out|T[\phi(x)\phi^\dagger(y)]|in\rangle_A}{\langle out|in\rangle_A} -iq\frac{\langle out|T[\phi(x)\phi^\dagger(y)]|in\rangle_A}{\langle out|in\rangle_A}(-\overleftarrow{\partial}_y^{\,\mu} - iqA(y))\right\}, \quad (1)$$

which is formally valid. Then from part a), show that

$$\langle out|in\rangle_A = \frac{C}{\det[m^2 - (\partial - iqA)^2]}.$$

Solution: First consider the variation

$$\begin{aligned}
&\delta \ln \mathrm{Det}([m^2 - (\partial - iqA)^2]^{-1}) = -\delta \mathrm{Tr} \ln [m^2 - (\partial - iqA)^2] \\
&= -\mathrm{Tr}[m^2 - (\partial - iqA)^2]^{-1} \delta[m^2 - (\partial - iqA)^2] \\
&= -\mathrm{Tr}[m^2 - (\partial - iqA)^2]^{-1} iq[\delta A \cdot (\partial - iqA) + (\partial - iqA) \cdot \delta A] \\
&= -iq \int d^4x \delta A_\mu(x) \lim_{y \to x} \left[(\partial - iqA)^\mu_x (x|[m^2 - (\partial - iqA)^2]^{-1}|y)\right. \\
&\quad \left. + (x|[m^2 - (\partial - iqA)^2]^{-1}|y)(-\overleftarrow{\partial} - iqA)^\mu_y\right] \\
&= -iq \int d^4x \delta A_\mu(x) \lim_{y \to x} \left[(\partial - iqA)^\mu_x i \frac{\langle out|T[\phi(x)\phi^\dagger(y)]|in\rangle_A}{\langle out|in\rangle_A}\right. \\
&\quad \left. + i \frac{\langle out|T[\phi(x)\phi^\dagger(y)]|in\rangle_A}{\langle out|in\rangle_A}(-\overleftarrow{\partial} - iqA)^\mu_y\right] \\
&= i \int d^4x \delta A_\mu(x) \frac{\langle out|j^\mu(x)|in\rangle_A}{\langle out|in\rangle_A}
\end{aligned}$$

Comparing to part a) we see that $\langle out|in\rangle_A$ and $\mathrm{Det}([m^2 - (\partial - iqA)^2]^{-1})$ satisfy identical first order variational equations and so must be the same up to a multiplicative constant independent of A.

c) Confirm the result of part b) to second order in A by using time dependent perturbation theory (*i.e.* by expanding the Dyson formula in interaction picture).

Solution:The interaction part of the Hamiltonian is in interaction picture

$$H' = \int d^3x [iqA^\mu(\phi_I^\dagger \nabla_\mu \phi_I - \nabla_\mu \phi_I^\dagger \phi_I) + q^2 \vec{A}^2 \phi_I^\dagger \phi_I]$$

Here we used $\pi_I = \dot{\phi}_I^\dagger$ because interaction picture fields satisfy the zero field equations of motion. Note that the term quadratic in A involves spatial components only. To second order the Dyson formula gives

$$\begin{aligned}
\ln\langle out|in\rangle &= 1 + q \int d^4x A^\mu \langle 0|(\phi_x^\dagger \partial_\mu \phi_x - \partial_\mu \phi_x^\dagger \phi_x)|0\rangle - iq^2 \int d^4x \vec{A}^2 \langle 0|\phi^\dagger \phi|0\rangle_c \\
&+ \frac{q^2}{2} \int d^4x d^4y \langle 0|T[A^\mu(x)(\phi_x^\dagger \partial_\mu \phi_x - \partial_\mu \phi_x^\dagger \phi_x) A^\nu(y)(\phi_y^\dagger \partial_\nu \phi_y - \partial_\nu \phi_y^\dagger \phi_y)]|0\rangle_c
\end{aligned} \tag{2}$$

where including only the connected diagrams computes $\ln\langle out|in\rangle$. On the other hand expanding $\ln \mathrm{Det}$ to the same order

$$\begin{aligned}
\ln \mathrm{Det}[m^2-(\partial-iqA)^2]^{-1} &= -\mathrm{Tr}\ln[m^2-(\partial-iqA)^2] \\
= -\mathrm{Tr}\ln[m^2-(\partial)^2] - \mathrm{Tr}\ln[1-[-iq\partial A-iqA\partial-q^2A^2](m^2-(\partial)^2)^{-1}] \\
= -\mathrm{Tr}\ln[m^2-(\partial)^2] + \mathrm{Tr}[-iq\partial A-iqA\partial-q^2A^2](m^2-(\partial)^2)^{-1} \\
+\frac{1}{2}\mathrm{Tr}[-iq\partial A-iqA\partial](m^2-(\partial)^2)^{-1}[-iq\partial A-iqA\partial](m^2-(\partial)^2)^{-1}
\end{aligned}$$

There is a rough match between these two expressions except that the first has a term $\vec{A}^2$ corresponding to $A_\mu A^\mu$ in the second; and the first has derivatives of fields inside the time ordering and the second has derivatives outside the time ordering. These two differences exactly cancel because of time derivatives of step functions. $\partial_0^x T[\phi(x)\phi^\dagger(y)] = T[\dot{\phi}(x)\phi^\dagger(y)]$ but

$$\partial_0^y\partial_0^x T[\phi(x)\phi^\dagger(y)] = T[\dot{\phi}(x)\dot{\phi}^\dagger(y)] - \delta(x^0-y^0)[\dot{\phi}(x),\phi^\dagger(y)] = T[\dot{\phi}(x)\dot{\phi}^\dagger(y)] + i\delta^4(x-y)$$

Thus if we write the Dyson expansion with derivatives outside the time ordering symbol, we then have to add the term $iq^2\int d^4x A_0^2\langle 0|\phi^\dagger\phi|0\rangle$ on the right. This completes the $\vec{A}^2 \to \vec{A}^2 - A_0^2 = A\cdot A$:

$$\begin{aligned}
\ln\langle out|in\rangle &= 1 + q\int d^4x A^\mu\langle 0|(\phi_x^\dagger\partial_\mu\phi_x - \partial_\mu\phi_x^\dagger\phi_x)|0\rangle - iq^2\int d^4x A_\mu A^\mu\langle 0|\phi^\dagger\phi|0\rangle_c \\
&+\frac{q^2}{2}\int d^4x d^4y \langle 0|T^*[A^\mu(x)(\phi_x^\dagger\partial_\mu\phi_x - \partial_\mu\phi_x^\dagger\phi_x)A^\nu(y)(\phi_y^\dagger\partial_\nu\phi_y - \partial_\nu\phi_y^\dagger\phi_y)]|0\rangle_c
\end{aligned}$$

where T^* means: take all derivatives outside of time ordering. Applying the Wick expansion, it is now easy to compare the two expansions and see that they are identical to second order.

45. In the vacuum polarization calculation we used the Feynman trick

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(Ax+B(1-x))^2} \tag{3}$$

to combine propagator denominators.

a) Prove this formula by direct integration.

Solution: The integral is just that of $[(A-B)x+B]^{-2}$ giving

$$-\frac{1}{A-B}[(A-B)x+B]^{-1}\Big|_o^1 = -\frac{1}{A-B}\left[\frac{1}{A}-\frac{1}{B}\right] = \frac{1}{AB} \tag{4}$$

Done

b) This is a special case of a formula combining n denominators

$$\frac{1}{A_1 A_2 \cdots A_n} = (n-1)! \int_0^1 dx_1 \cdots dx_n \delta(1 - \sum_k x_k) \frac{1}{[\sum_k x_k A_k]^n} \tag{5}$$

Prove this generalization. A good first step is to write $1/A_k = \int_0^\infty dT_k e^{-T_k A_k}$. then change variables to $T = \sum_k T_k$ and $x_k = T_k/T$. Then do the integral over T.

Solution: Startiung with the hint:

$$\begin{aligned}
\frac{1}{A_1 A_2 \cdots A_n} &= \int_0^\infty dT_1 \cdots dT_n e^{-\sum_{k=1}^n A_k T_k} \\
&= \int_0^\infty dT dT_1 \cdots dT_n e^{-\sum_{k=1}^n A_k T_k} \delta(T - \sum_k T_k) \\
&= \int_0^\infty dT T^n dx_1 \cdots dx_n e^{-T \sum_{k=1}^n A_k x_k} \delta(T[1 - - \sum_k x_k]) \\
&= \int_0^\infty dT T^{n-1} dx_1 \cdots dx_n e^{-T \sum_{k=1}^n A_k x_k} \delta(1 - \sum_k x_k)
\end{aligned} \tag{6}$$

The T integral is then

$$\int_0^\infty dT T^{n-1} e^{-T \sum_{k=1}^n A_k x_k} = \frac{1}{[\sum_{k=1}^n A_k x_k]^n} \int dt t^{n-1} e^{-t} = (n-1)! \frac{1}{[\sum_{k=1}^n A_k x_k]^n} \tag{7}$$

which completes the proof. The integral in the last line is just the representation of the Euler gamma function $\Gamma(n)$.

46. **Integration over Euclidean D dimensional spacetime.**

a) Confirm the validity of the replacement shown in Eq (10.28) of the lecture notes.

Solution:

$$\begin{aligned}
N^{\mu\nu}(p + xk, k) &= 8(p + xk)^\mu (p + xk)^\nu - 4((p + xk)^\mu k^\nu + (p + xk)^\nu k^\mu) \\
&\quad -4\eta^{\mu\nu}(m^2 + (p + xk) \cdot (p - (1 - x)k)) \\
&\to 8(p^\mu p^\nu + x^2 k^\mu k^\nu) - 8x k^\mu k^\nu - 4\eta^{\mu\nu}(m^2 + p^2 - x(1 - x)k^2) \\
&\to \eta^{\mu\nu} \left(\frac{8}{D} - 4 \right) p^2 + 4x(1 - x)[k^2 \eta^{\mu\nu} - 2k^\mu k^\nu] - 4\eta^{\mu\nu} m^2 \\
&\to -2\eta^{\mu\nu} p^2 + 4x(1 - x)[k^2 \eta^{\mu\nu} - 2k^\mu k^\nu] - 4\eta^{\mu\nu} m^2
\end{aligned} \tag{8}$$

In the second line we dropped terms linear in p and in the third averaged over angles in D dimensions.

b) Derive equation (10.32) in our lecture notes, following the suggestions in the text.

Solution: We calculate $\int d^D pe^{-p^2}$ in two different ways. First doing D cartesian integrals of a gaussian gives $\pi^{D/2}$. In teh second we use polar coorinates

$$\int d^D pe^{-p^2} = \int d\Omega_D \int_0^\infty dpp^{D-1}e^{-p^2} = \Omega_D \int_0^\infty du \frac{1}{2} u^{(D-2)?2} e^{-u} = \Omega_D \frac{\Gamma(D/2)}{2} \quad (9)$$

Setting the two evaluations equal gives (10.32):

$$\Omega_D = \frac{2\pi^{D?@}}{\Gamma(D/2)} \quad (10)$$

c) Derive equation (10.33), using the integral representation for the Euler beta function

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 dtt^{x-1}(1-t)^{y-1}. \quad (11)$$

You will have to scale out A and do an appropriate change of integration variables in polar coordinates after the angular integration.

Solution: We start in polar coordinates

$$\begin{aligned}
\int d^D p \frac{(p^2)^m}{(p^2+A^2)^n} &= \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty dp \frac{p^{D-1+2m}}{(p^2+A^2)^n} = \frac{2\pi^{D/2}}{\Gamma(D/2)} A^{D+2m-2n} \int_0^\infty dp \frac{p^{D-1+2m}}{(p^2+1)^n} \\
&= \frac{2\pi^{D/2}}{\Gamma(D/2)} A^{D+2m-2n} \int_1^\infty \frac{du}{2} \frac{(u-1)^{D/2-1+m}}{u^n} \\
&= \frac{2\pi^{D/2}}{\Gamma(D/2)} A^{D+2m-2n} \int_0^1 \frac{dv}{2} v^{n-m-1-D/2}(1-v)^{D/2-1+m} \\
&= \frac{A^{D+2m-2n}\pi^{D/2}}{\Gamma(D/2)} \frac{\Gamma(m+D/2)\Gamma(n-m-D/2)}{\Gamma(n)} \quad (12)
\end{aligned}$$

In the first line we scaled out A with the variable change $p \to Ap$, in the second line we changed variables to $u = p^1 + 1$, and in the third line changed to $v = 1/u$.

47. Instead of using the cutoff Λ to do the divergent integrals, equations (10.35)and (10.37), use the results proved in the previous problem to do the integrals

$$\int \frac{d^D p}{(2\pi)^D} \frac{1}{[p^2+C^2]^2}, \qquad \int \frac{d^D p}{(2\pi)^D} \frac{p^2}{[p^2+C^2]^2} \quad (13)$$

for general Euclidean spacetime dimension D. After the angular integrals, the integrals over the magnitude p will converge for sufficiently small $D < 4$. Now consider your results in the limit $D \to 4$, and compare the result to that using the cutoff Λ described in the notes. You will find poles at $D = 4$ instead of $\ln \Lambda$'s.

Solution: For the first integral we use the previous problem witn $A = C$, $n = 2$, and $m = 0$ to get

$$\int \frac{d^D p}{(2\pi)^D} \frac{1}{[p^2 + C^2]^2} = \frac{C^{D-4}\pi^{D/2}}{\Gamma(D/2)} \frac{\Gamma(D/2)\Gamma(2-D/2)}{(2\pi)^D \Gamma(2)} = C^{D-4} \frac{\Gamma(2-D/2)}{(4\pi)^{D/2}} \tag{14}$$

For the second integral we put $m = 1$ to get

$$\int \frac{d^D p}{(2\pi)^D} \frac{p^2}{[p^2 + C^2]^2} = \frac{C^{D-2}\pi^{D/2}}{\Gamma(D/2)} \frac{\Gamma(1+D/2)\Gamma(1-D/2)}{(2\pi)^D \Gamma(2)} = C^{D-2} \frac{(D/2)\Gamma(2-D/2)}{(1-D/2)(4\pi)^{D/2}} \tag{15}$$

Putting these results into the expression for $T^{\mu\nu}(k)$ gives

$$\begin{aligned} T_D^{\mu\nu}(k) &= -Q^2 \frac{\Gamma(2-D/2)}{(4\pi)^{D/2}} \int_0^1 dx C^{D-4} [(8/D-4)/(2/D-1)\eta^{\mu\nu}(m^2 + x(1-x)k^2) \\ &\quad +4x(1-x)(k^2\eta^{\mu\nu} - 2k^\mu k^\nu) - 4m^2\eta^{\mu\nu}] \\ &= -Q^2 \frac{\Gamma(2-D/2)}{(4\pi)^{D/2}} \int_0^1 dx C^{D-4} [8x(1-x)(k^2\eta^{\mu\nu} - k^\mu k^\nu)] \end{aligned} \tag{16}$$

The answer is properly transverse (gauge invariant) for all D. $\Gamma(z)$ has pokes at $z = 0, -1, -2, \cdots$. Therefore as $D \to 3$,

$$\begin{aligned} \Gamma(2-D/2) &= \frac{\Gamma(3-D/2)}{2-D/2} \sim \frac{\Gamma(1) + (2-D/2)\Gamma'(1)}{2-D/2)} \sim \frac{2}{4-D} + \Gamma'(1) \\ \Gamma(2-D/2)C^{D-4} &\sim \left[\frac{2}{4-D} + \Gamma'(1)\right][1 + (D-4)\ln C] \sim \frac{2}{4-D} + \Gamma'(1) - \ln C^2 + O(D-4) \\ &\sim \frac{2}{4-D} + \Gamma'(1) - \ln(m^2 + x(1-x)k^2) + O(D-4) \end{aligned} \tag{17}$$

Inserting into $T^{\mu\nu}$, we find

$$\begin{aligned} T_D^{\mu\nu}(k) &= (k^\mu k^\nu - k^2\eta^{\mu\nu}) T_D(k^2) \\ T_D(k^2) &\sim \frac{8Q^2}{(4\pi)^{D/2}} \int_0^1 dx x(1-x) \left[\frac{2}{4-D} + \Gamma'(1) - \ln(m^2 + x(1-x)k^2)\right] \end{aligned} \tag{18}$$

for $D \sim 4$. The two integrals of this problem behave as $D \to 4$ as:

$$\begin{aligned} \int \frac{d^D p}{(2\pi)^D} \frac{1}{[p^2 + C^2]^2} &\sim \frac{1}{(4\pi)^{D/2}} \left[\frac{2}{4-D} + \Gamma'(1) - \ln C^2\right] \\ \int \frac{d^D p}{(2\pi)^D} \frac{p^2}{[p^2 + C^2]^2} &\sim \frac{C^2}{(2/D-1)(4\pi)^{D/2}} \left[\frac{2}{4-D} + \Gamma'(1) - \ln C^2\right] \end{aligned} \tag{19}$$

As expected the divergence appears as a pole at $D = 4$.