Quantum Field Theory

Problem Set 13

Due: 24 April 2019 (Last day of class)

Reading: Lecture Notes Chapters 11-13.

48. **Vacuum Polarization contribution to the Lamb shift**. We derived the relation between D and E, which after charge renormalization, to first order in e^2 is

$$\tilde{E}(k) = \left[1 + \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left\{1 + x(1-x)\frac{k^2}{m^2}\right\}\right]\tilde{D}(k)$$

In consequence, the potential energy of the electron in a hydrogen atom is modified from the lowest order $-e^2/4\pi|\mathbf{x}|$. In this problem you are to calculate the effect of this modification on the hydrogen energy levels, by inserting the change of the potential into lowest order non-relativistic perturbation theory.

a) Explain why values of $k^2/m^2 \ll 1$ as well as $k^{02} \ll \vec{k}^2$ are involved in this calculation.

Solution: We know that the speed of the electron in hydrogen is $v \sim \alpha = 1/137 \ll 1$. Thus its momentum is $O(m\alpha) \ll m$ and its kinetic energy is $O(m\alpha^2) \ll m\alpha$. Since the perturbation will be evaluated with electron wave functions these will be the sizes of momentum and energy transfers respectively.

b) The estimates of part a) justify expanding the logarithm in the formula to estimate the integral. Show that the extra potential energy is of the form $-V_1\delta^3(\vec{x})$ and find the constant V_1.

Solution: Making these approximations we have

$$\tilde{E}(k) \approx \left[1 + \frac{e^2}{2\pi^2}\int_0^1 dx\, x^2(1-x)^2\frac{\vec{k}^2}{m^2}\right]\tilde{D}(k)$$

which we can transform to coordinate space as

$$E(\vec{x}) \approx \left[1 + \frac{e^2}{2\pi^2}\int_0^1 dx\, x^2(1-x)^2\frac{-\nabla^2}{m^2}\right]D(\vec{x})$$

In terms of the electrostatic potential of the proton this relation reads

$$A^0_{TOT}(\vec{x}) \approx \left[1 + \frac{e^2}{2\pi^2}\int_0^1 dx\, x^2(1-x)^2\frac{-\nabla^2}{m^2}\right]\frac{e}{4\pi|\vec{x}|} = \frac{e}{4\pi|\vec{x}|} + \frac{e^3}{60\pi^2 m^2}\delta(\vec{x})$$

So that the electron sees a potential energy

$$V(r) = -eA^0 = -\frac{e^2}{4\pi|\vec{x}|} - \frac{e^4}{60\pi^2 m^2}\delta(\vec{x})$$

Thus the extra potential energy is of the form $-V_1\delta^3(\vec{x})$ with $V_1 = 4\alpha^2/(15m^2)$.

c) Explain why only s levels ($l = 0$) will be affected by this perturbation.

Solution: The energy shifts from this perturbation are therefore just $\Delta E_{nl} = -V_1|\psi_{nl}(0)|^2$. This is zero for $l \neq 0$ so only the S states shift.

d) Calculate the shift in the $2s$ state in electron-volts. (Remember that we have put $\hbar = c = 1$. Restore them at some late stage to get dimensions right.)

Solution:$\Delta E_{n00} = -V_1 R_{n0}^2(0)/4\pi$. Recall that

$$R_{2s}(r) = (1 - r/2a_0)e^{-r/2a_0}/\sqrt{2a_0^3}$$

where $a_0 = 1/(m\alpha)$ is the Bohr radius. Thus $\Delta E_{20} = -V_1/(8\pi a_0^3) = -m\alpha^5/(30\pi) = -\mathrm{Ry}\alpha^3/(15\pi) \approx -1.1 \times 10^{-7}\mathrm{eV}$.

Note that this is a relatively small contribution to the Lamb shift: The actual Lamb shift $E_{2s} - E_{2p} \approx 4 \times 10^{-6}\mathrm{eV}$ is an order of magnitude larger and of opposite sign. The bulk of the effect comes from the radiative corrections to the vertex function. But the precision of measurements is such that the vacuum polarization effect we just calculated must be included to get agreement with experiment.

49. Path integral expression for the Partition function

It is sometimes interesting to interpret the imaginary time path integral directly as giving the statistical mechanics of the system at non-zero temperature. The partition function $Z(\beta)$ at temperature β^{-1} is defined as $Tr\ e^{-\beta H} = \int dq\langle q|e^{-\beta H}|q\rangle$, where q is a complete set of coordinates. But we have an expression for $\langle q''|e^{-\beta H}|q'\rangle$ as an integral over all paths from q' at $t = 0$ to q'' at $t = -i\beta$. So we get $Z(\beta) = \int Dq \exp\{i\int_0^\beta L(-i\tau)(-i)d\tau\}$ where the integral is over all paths with $q(-i\beta) = q(0)$. For the case of one degree of freedom with $L = \frac{1}{2}m\dot{q}^2 - V(q)$, this becomes

$$Z(\beta) = \int Dq \exp\left[-\int_0^\beta \left\{\frac{1}{2}m\left(\frac{dq}{d\tau}\right)^2 + V(q)\right\}d\tau\right], \qquad q(\beta) = q(0)$$

To define the integral, we break the interval 0 to β into N parts of length ϵ and replace $\int (dq/d\tau)^2\, d\tau$ by $(1/\epsilon)\sum_{r=1}^N (q_{r+1} - q_r)^2$ (with $q_{N+1} \equiv q_1$), $\int V(q)d\tau$ by $\epsilon\sum_{r=1}^N V(q_r)$, and then evaluate $(m/2\pi\epsilon)^{N/2}\int dq_1 \ldots dq_N$. In practice we usually calculate ratios of path integrals or fix normalizations using closure, thus avoiding an explicit evaluation of the path integral. In this exercise we shall calculate $Z(\beta)$ directly for the harmonic oscillator, $V(q) = \frac{1}{2}m\omega^2 q^2$.

a) Show that the formula for Z takes the form

$$Z(\beta) = \left(\frac{m}{2\pi\epsilon}\right)^{N/2} \int \prod_{r=1}^{N} dq_r \exp\left[-\frac{1}{2}\sum_{r,s=1}^{N} A_{rs} q_r q_s\right].$$

finding an explicit expression for the $N \times N$ matrix A_{rs}.

Solution: The discretized imaginary time action is

$$\frac{m}{2\epsilon}\sum_{r=1}^{N}(q_{r+1}-q_r)^2 + \frac{m\omega^2\epsilon}{2}\sum_{r=1}^{N} q_r^2 = \frac{1}{2}\sum_{r,s} q_r \left[\delta_{rs}\left(\frac{2m}{\epsilon}+m\omega^2\epsilon\right) - \frac{m}{\epsilon}(\delta_{r,s+1}+\delta_{r+1,s})\right] q_s$$

So

$$A_{rs} = \delta_{rs}\left(\frac{2m}{\epsilon}+m\omega^2\epsilon\right) - \frac{m}{\epsilon}(\delta_{r,s+1}+\delta_{r+1,s})$$

b) We can always change variables to linear combinations of q_r which makes A_{rs} diagonal. Show that $Z(\beta) = (m/\epsilon)^{N/2}\prod_{\nu=1}^{N}\lambda_\nu^{-1/2}$ where λ_ν are the eigenvalues of the equations $\sum\limits_s A_{rs}q_s = \lambda q_r$. (In fact $\prod \lambda_\nu = \det A$.)

Solution:The transformation to normal coordinates is an orthogonal matrix so the Jacobian for the change of integration variables is unity. Thus

$$Z(\beta) = \left(\frac{m}{2\pi\epsilon}\right)^{N/2} \int \prod_{r=1}^{N} dq_r \exp\left[-\frac{1}{2}\sum_{\nu=0}^{N-1}\lambda_\nu q_r^2\right] = \left(\frac{m}{2\pi\epsilon}\right)^{N/2}\prod_\nu\left(\frac{2\pi}{\lambda_\nu}\right)^{1/2} = \left(\frac{m}{\epsilon}\right)^{N/2}\prod_\nu \lambda_\nu^{-1/2}.$$

c) Show that $\lambda_\nu = 2(m/\epsilon)(1-\cos\theta_\nu) + m\epsilon\omega^2$, $\theta_\nu = 2\nu\pi/N$, $\nu = 0,\ldots,N-1$. (Remember $q_{N+1} \equiv q_1$.) This is essentially the same mathematical problem as finding the normal modes of a chain of identical masses connected by identical springs. Remember your classical mechanics!

Solution:Making the ansatz that the eigenvectors of A_{rs} are of the form $v_r = Ne^{i\theta r}$, we easily read off the eigenvalue $\lambda(\theta) = \frac{2m}{\epsilon}(1-\cos\theta)+m\omega^2\epsilon$. The condition $v_1 = v_{N+1}$ then implies $e^{iN\theta} = 1$, which has solutions $\theta_\nu = 2\pi\nu/N$, where ν is an integer. Restricting $\nu = 0, 1, \ldots, N-1$ gives N distinct eigenvalues, which proves that these eigenvectors are complete since A is an $N \times N$ symmetric matrix.

d) Evaluate $Z(\beta)$ using the identity $2(\cos N\theta - 1) = \prod\limits_{\nu=0}^{N-1}(2\cos\theta - 2\cos\theta_\nu)$ (which is true because $\cos N\theta$ is a polynomial in $\cos\theta$ and the R.H.S. has the same zeros as the L.H.S. and the right coefficient of $\cos^N\theta$). Verify that it agrees with a direct evaluation of $\mathrm{Tr}e^{-\beta H}$ using standard raising and lowering operators $a, a^\dagger$.

Solution:We have $\prod_\nu \lambda^\nu = \left(\frac{m}{\epsilon}\right)^N \prod_\nu(2(1-\cos\theta_\nu)+\omega^2\epsilon^2)$ from which we infer that θ must satisfy $\cos\theta = 1+\omega^2\epsilon^2/2$. For $\epsilon\to 0$, we learn that $-\theta^2\to\omega^2\epsilon^2$ or $\theta\to i\omega\epsilon$ so $N\theta\to i\omega\beta$. Thus

$$\prod_\nu \lambda^\nu = \left(\frac{m}{\epsilon}\right)^N 2(\cosh(\omega\beta)-1) = \left(\frac{m}{\epsilon}\right)^N 4(\sinh^2(\omega\beta/2)$$

From which it follows that $Z(\beta) = 1/(2\sinh(\omega\beta/2)) = e^{-\beta\omega/2}(1-e^{-\beta\omega})^{-1}$, the known answer. This is obtained by the formula $Z=\sum_n e^{-\beta E_n} = \sum_{n=0}^\infty e^{-\beta\omega(n+1/2)}$.

Note that we could interpret this calculation as the partition function of an elastic string of length β in a potential V in classical statistical mechanics.

50. **Compton Scattering**: In this problem and the next (Problem 51), you will calculate the differential cross section for a photon scattering off an electron at rest. In this problem you are to derive the squared amplitude. According to our discussion of the free quantum field for the photon, its polarization is specified by a complex three vector, $\boldsymbol{\epsilon}$ perpendicular to its momentum $\boldsymbol{k}$. it is convenient to denote this as a four vector ϵ_μ with time component $\epsilon_0 = 0$. Then it follows that an incoming (outgoing) photon corresponds to a factor of ϵ (ϵ^*) dotted into the vertex it enters.(leaves).

a) Write down the Feynman amplitude for this process in terms of Dirac spinors for the initial and final electrons and the polarization vectors $\epsilon_\mu(k)$ and $\epsilon'_\mu(k')$ for the initial and final photons. Show that the change $\epsilon_\mu\to\epsilon_\mu + Ck_\mu$ leaves the amplitude unchanged. This verifies gauge invariance, and provides the flexibility to use a polarization vector with $\epsilon_0\neq 0$, provided $k^\mu\epsilon_\mu = 0$.

Solution: There are two diagrams contributing at tree level.

$$iQ^2\bar{u}'\gamma\cdot\epsilon'^*\frac{m-\gamma\cdot(p+k)}{m^2+(p+k)^2}\gamma\cdot\epsilon u, \qquad iQ^2\bar{u}'\gamma\cdot\epsilon\frac{m-\gamma\cdot(p-k')}{m^2+(p-k')^2}\gamma\cdot\epsilon'^* u \tag{1}$$

Combining the two diagrams gives

$$\mathcal{M} = \frac{iQ^2}{2p\cdot kp\cdot k'}\bar{u}'\left[p\cdot k'\gamma\cdot\epsilon'^*(m-(p+k)\cdot\gamma)\gamma\cdot\epsilon - p\cdot k\gamma\cdot\epsilon(m-(p-k')\cdot\gamma)\gamma\cdot\epsilon'^*\right]u$$

To verify gauge invariance we replace ϵ_μ by k_μ:

$$\begin{aligned}
&\bar{u}'\gamma\cdot\epsilon'^*\frac{m-\gamma\cdot(p+k)}{m^2+(p+k)^2}\gamma\cdot ku + \bar{u}'\gamma\cdot k\frac{m-\gamma\cdot(p-k')}{m^2+(p-k')^2}\gamma\cdot\epsilon'^* u\\
&= \bar{u}'\gamma\cdot\epsilon'^*\frac{m-\gamma\cdot(p+k)}{m^2+(p+k)^2}(\gamma\cdot(k+p)+m)u + \bar{u}'\gamma\cdot(p'+k'-p)\frac{m-\gamma\cdot(p-k')}{m^2+(p-k')^2}\gamma\cdot\epsilon'^* u\\
&= \bar{u}'\gamma\cdot\epsilon'^* u + \bar{u}'(-m+\gamma\cdot(k'-p))\frac{m-\gamma\cdot(p-k')}{m^2+(p-k')^2}\gamma\cdot\epsilon'^* u = \bar{u}'\gamma\cdot\epsilon'^* u - \bar{u}'\gamma\cdot\epsilon'^* u = 0,
\end{aligned}$$

where in the second and third lines we used the Dirac equation $(m+\gamma\cdot p)u = 0$ or $\bar{u}'(m+\gamma\cdot p') = 0$.

b) Express the squared amplitude summed over final electron spins and averaged over initial electron spins as traces of products of gamma matrices.

Solution: Call A the spinor matrix element in b) so we can write

$$\sum_{\lambda,\lambda'} |A|^2 = \mathrm{Tr}\,[(p\cdot k'\gamma\cdot\epsilon'^*(m-(k+p)\cdot\gamma)\gamma\cdot\epsilon - p\cdot k\gamma\cdot\epsilon(m-(p-k')\cdot\gamma)\gamma\cdot\epsilon'^*)(m-\gamma\cdot p)$$
$$(p\cdot k'\gamma\cdot\epsilon^*(m-(p+k)\cdot\gamma)\gamma\cdot\epsilon' - p\cdot k\gamma\cdot\epsilon'(m-(p-k')\cdot\gamma)\gamma\cdot\epsilon^*)(m-\gamma\cdot p')]$$

Then

$$\sum_{\lambda,\lambda'} |\mathcal{M}|^2 = \frac{Q^4}{4(p\cdot kp\cdot k')^2}\sum_{\lambda,\lambda'}|A|^2 \tag{2}$$

The trace involves up to 8 gamma matrices!

c) Sum over final and average over initial photon polarizations, using the results for the **spatial** components. (The time components are zero!).

$$\sum_{pol} \epsilon_l \epsilon_m^* = \delta_{lm} - \frac{k_l k_m}{\boldsymbol{k}^2}, \tag{3}$$

and similarly for ϵ'.

Solution: It is helpful to express the polarization sum in covariant notation. For that purpose introduce the 4 vector $n^\mu = (1,0,0,0)$. Then we can write

$$\sum_{pol} \epsilon_\mu \epsilon_\nu^* = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{\boldsymbol{k}^2} + \frac{n_\mu k_\nu + n_\nu k_\mu}{|\boldsymbol{k}|}, \tag{4}$$

It is easy to confirm that the right side is zero if $\mu = 0$ and/or $\nu = 0$, and the last two terms vanish when μ, ν are both spatial. When this identity is used in the squared matrix element, the terms with a k_μ and/or k_ν can be dropped because they correspond to ϵ or ϵ' being replaced by k or k' in either $\mathcal{M}$ or $\mathcal{M}^*$. Thus we can write

$$\sum_{\lambda,\lambda',pol} |A|^2 = \mathrm{Tr}\left[(p\cdot k'\gamma^\mu(m-(k+p)\cdot\gamma)\gamma^\lambda - p\cdot k\gamma^\lambda(m-(p-k')\cdot\gamma)\gamma^\mu)(m-\gamma\cdot p)\right.$$
$$\left.(p\cdot k'\gamma_\lambda(m-(p+k)\cdot\gamma)\gamma_\mu - p\cdot k\gamma_\mu(m-(p-k')\cdot\gamma)\gamma_\lambda)(m-\gamma\cdot p')\right]$$

d) Evaluate the resulting traces and confirm that the result matches the one quoted in Srednicki Eq. (11.50).

Solution: There are four distinct traces to evaluate: First take the product of the first terms in each factor:

$$\begin{aligned}
&\mathrm{Tr}\left[(p\cdot k'\gamma^\mu(m-(k+p)\cdot\gamma)\gamma^\lambda)(m-\gamma\cdot p)(p\cdot k'\gamma_\lambda(m-(p+k)\cdot\gamma)\gamma_\mu(m-\gamma\cdot p')\right]\\
&=(p\cdot k')^2\mathrm{Tr}\left[((m-(k+p)\cdot\gamma))(-4m-2\gamma\cdot p)((m-(p+k)\cdot\gamma)(-4m-2\gamma\cdot p')\right]\\
&=(p\cdot k')^2\left[64m^4+64m^2(p+p')\cdot(p+k)-16m^2p\cdot p'-64m^2(p+k)^2\right.\\
&\left.+16(2p\cdot(p+k)p'\cdot(p+k)-p\cdot p'(p+k)^2\right]\\
&=4(m^2-u)^2[4m^4+4m^2(-m^2-s)+m^2(s+u)/2+4m^2s+(m^2+s)^2/2-s(s+u)/2]\\
&=2(m^2-u)^2[m^2(3s+u)+m^4-su] \qquad (5)
\end{aligned}$$

where we used $p\cdot k=[(p+k)^2+m^2]/2=(m^2-s)/2$, $p\cdot k'=-[(p-k')^2+m^2]/2=-(m^2-u)/2$, $p\cdot(p+k)=-m^2+(m^2-s)/2=-(m^2+s)/2=p'\cdot(p'+k')=p'\cdot(p+k)$, and $p\cdot p'=-(p-p')^2/2-m^2=t/2-m^2=-(s+u)/2$.

The product of the second terms in each factor is related to this by $k\leftrightarrow -k'$ or $s\leftrightarrow u$.

$$2(m^2-s)^2[m^2(3u+s)+m^4-su] \qquad (6)$$

The remaining terms are the product of the first term in the first factor by the second term in the second factor and the product of the second term in the first factor by the first term in the second factor The first of these is

$$\begin{aligned}
&-k\cdot pk'\cdot p\mathrm{Tr}\left[\gamma^\mu(m-(k+p)\cdot\gamma)\gamma^\lambda(m-\gamma\cdot p)\gamma_\mu(m-(p-k')\cdot\gamma)\gamma_\lambda(m-\gamma\cdot p')\right]\\
&=-k\cdot pk'\cdot p\mathrm{Tr}\left[\gamma^\mu(m^2\gamma^\lambda-m(k+p)\cdot\gamma\gamma^\lambda-m\gamma^\lambda\gamma\cdot p+(k+p)\cdot\gamma\gamma^\lambda\gamma\cdot p)\gamma_\mu(m-(p-k')\cdot\gamma)\gamma_\lambda(m-\gamma\cdot p')\right]\\
&=-k\cdot pk'\cdot p\mathrm{Tr}\left[(2m^2\gamma^\lambda-4m(k+2p)^\lambda+2p\cdot\gamma\gamma^\lambda\gamma\cdot(k+p))(m-(p-k')\cdot\gamma)\gamma_\lambda(m-\gamma\cdot p')\right]\\
&=4mk\cdot pk'\cdot p\mathrm{Tr}\left[(m-(p-k')\cdot\gamma)(k+2p)\cdot\gamma(m-\gamma\cdot p')\right]\\
&-k\cdot pk'\cdot p\mathrm{Tr}\left[(2m^2\gamma^\lambda+2p\cdot\gamma\gamma^\lambda\gamma\cdot(k+p))(m-(p-k')\cdot\gamma)\gamma_\lambda(m-\gamma\cdot p')\right]\\
&=16m^2k\cdot pk'\cdot p\left[(k+2p)\cdot(p'+p-k')\right]\\
&-k\cdot pk'\cdot p\mathrm{Tr}\left[(-8m^3-4m^2\gamma\cdot p'+4mp\cdot\gamma\gamma\cdot(k+p)-8p\cdot p'\gamma\cdot(k+p))(m-(p-k')\cdot\gamma))\right]\\
&=16m^2k\cdot pk'\cdot p\left[(k+2p)\cdot(2p'-k)\right]\\
&-k\cdot pk'\cdot p\left[-32m^4-16m^2(p-k')\cdot p'-16m^2p\cdot(k+p)-32p\cdot p'(k+p)\cdot(p-k')\right] \qquad (7)
\end{aligned}$$

. To express this in terms of s,u we need

$$\begin{aligned}
(2p+k)\cdot(2p'-k) &= 4p\cdot p'+2k\cdot(p'-p)=-2(s+u)+2k\cdot(k-k')=-u-s-2m^2\\
(p+k)\cdot(p-k') &= -m^2-k\cdot k'+p\cdot(k-k')=-m^2\\
p'\cdot(p-k') &= p'\cdot(p'-k)=-m^2+\frac{1}{2}(p'-k)^2+m^2/2=\frac{-m^2-u}{2} \qquad (8)
\end{aligned}$$

Inserting these values in the last two lines of the previous equation gives

$$\begin{aligned}
&-k\cdot pk'\cdot p\left[16m^2(u+s+2m^2)-32m^4+8m^2(m^2+u)+8m^2(m^2+s)-16m^2(u+s)\right]\\
&=8m^2k\cdot pk'\cdot p\left[-2m^2-u-s\right]
\end{aligned}$$

The remaining term is obtained from this by the interchange $s \leftrightarrow u$ which just doubles this one. Putting everything together we have

$$\sum_{\lambda,\lambda',pol} |A|^2 = 2(m^2-s)^2[m^2(3u+s)+m^4-su]+2(m^2-u)^2[m^2(3s+u)+m^4-su]$$
$$+4m^2(s-m^2)(m^2-u)\left[-2m^2-u-s\right] \quad (9)$$

The squared Feynman amplitude summed over all spins and polarizations is this expression times $Q^4/(4(p\cdot k)^2(p\cdot k')^2) = 4Q^2/[(s-m^2)^2(u-m^2)^2]$. In addition we need to divide by 4 to convert the sum over initial spin and polarization to averages. Then we have finally

$$\frac{1}{4}\sum_{\lambda,\lambda',pol} |\mathcal{M}|^2 = 2Q^4\left[\frac{m^2(3u+s)+m^4-su}{(u-m^2)^2}+\frac{m^2(3s+u)+m^4-su}{(s-m^2)^2}\right.$$
$$\left.-2m^2\frac{2m^2+u+s}{(s-m^2)(m^2-u)}\right] \quad (10)$$

in agreement with S, Problem 11.2 after putting $Q=-e$, so $2Q^4 = 2e^4 = 32\pi^2\alpha^2$.

51. S, Problem 11.2. This exercise guides you through the cross section evaluation for Compton scattering starting from the result obtained is part d) of the previous problem.

Solution:

a)$s = -(p+k)^2 = m_e^2 + 2m_e\omega$, and $u = -(p-k')^2 = m_e^2 - 2m_e\omega'$, where we used the fact that the electron momentum $p = (m_e, \mathbf{0})$ in the lab system.

b)$t = -(k-k')^2 = -2\omega\omega'(1-\cos\theta)$, but also $t = 2m_e^2 - s - u = 2m_e(\omega_\prime - \omega)$. Thus

$$\cos\theta = 1 - \frac{m_e(\omega-\omega')}{\omega\omega'}$$

It is interesting to also write this relation as

$$\omega' = \frac{m\omega}{m+\omega(1-\cos\theta)}$$

which shows the angular dependence of the final photon frequency.

c)Plugging into $|\mathcal{T}|^2$ the expressions for s,u yields,

$$|\mathcal{T}|^2 = 32\pi^2\alpha^2\left[\tfrac{m^2+m\omega+\omega\omega'}{\omega^2}+\tfrac{m^2-m\omega'+\omega\omega'}{\omega'^2}-\tfrac{2m^2+m(\omega-\omega')}{\omega\omega'}\right] \quad (11)$$
$$= 32\pi^2\alpha^2\left[m^2\left(\tfrac{1}{\omega}-\tfrac{1}{\omega'}\right)^2+2m\left(\tfrac{1}{\omega}-\tfrac{1}{\omega'}\right)+\tfrac{\omega}{\omega'}+\tfrac{\omega'}{\omega}\right]$$
$$= 32\pi^2\alpha^2\left[\tfrac{\omega}{\omega'}+\tfrac{\omega'}{\omega}-\sin^2\theta\right] \quad (12)$$

The velocity of the photon is 1, and the phase space integral over the electron momentum and final photon energy yields the factor

$$\frac{\omega' d\Omega}{64\pi^2 m\omega(m + \omega(1 - \cos\theta))} = \frac{\omega'^2 d\Omega}{64\pi^2 \omega^2 m^2}$$

Multiplying these expressions together yields (11.51).

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \omega'^2}{2\omega^2 m^2} \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta \right]$$