Quantum Field Theory I

Problem Set 1: Solutions

Due: 5 September 2007

1. S, 2.8

a) $U^{-1}\phi U \sim \phi + \frac{i}{2}\delta\omega_{\mu\nu}[\phi, M^{\mu\nu}]$ and $\phi(x_\mu - \delta\omega_{\mu\nu}x^\nu) \sim \phi - \delta\omega_{\mu\nu}x^\nu\partial_\mu\phi$. Comparing we find

$$[\phi, M^{\mu\nu}] = i(x^\nu\partial_\mu - x^\mu\partial_\nu)\phi = -i\mathcal{L}^{\mu\nu}.$$

b) Apply a) a second time.

c) Write out and cancel terms in pairs.

d)

$$[\phi, [M^{\mu\nu}, M^{\rho\sigma}]] = -[M^{\mu\nu}, [M^{\rho\sigma}, \phi]] - [M^{\rho\sigma}, [\phi, M^{\mu\nu}]] = -\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\phi + \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\phi$$

e)

$$\begin{aligned}\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} &= -x^\mu\partial^\nu x^\rho\partial^\sigma + x^\mu\partial^\nu x^\sigma\partial^\rho + x^\nu\partial^\mu x^\rho\partial^\sigma - x^\nu\partial^\mu x^\sigma\partial^\rho \\ &= x^\mu x^\rho\partial^\nu\partial^\sigma + x^\mu x^\sigma\partial^\nu\partial^\rho + x^\nu x^\rho\partial^\mu\partial^\sigma - x^\nu x^\sigma\partial^\mu\partial^\rho \\ &\quad - \eta^{\nu\rho}x^\mu\partial^\sigma + \eta^{\nu\sigma}x^\mu\partial^\rho + \eta^{\mu\rho}x^\nu\partial^\sigma - \eta^{\mu\sigma}x^\nu\partial^\rho\end{aligned}$$

In subtracting $\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}$ from this all the terms with 2 x's and 2 ∂'s cancel, and the remaining terms are linear in the $\mathcal{L}$'s.

f) From d).e)

$$\begin{aligned}[\phi, [M^{\mu\nu}, M^{\rho\sigma}]] &= (-\eta^{\nu\rho}i\mathcal{L}^{\mu\sigma} + \eta^{\nu\sigma}i\mathcal{L}^{\mu\rho} + \eta^{\mu\rho}i\mathcal{L}^{\nu\sigma} - \eta^{\mu\sigma}i\mathcal{L}^{\nu\rho})\phi \\ &= [\phi, i(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}iM^{\mu\sigma} + \eta^{\nu\sigma}M^{\mu\rho})]\end{aligned}$$

so the commutator of (2.16) with ϕ holds.

2. S, 2.9.

a) $(I + \delta\omega)^\mu{}_\rho\partial^\rho\phi(x - \delta\omega x) \sim \partial^\mu\phi(x) - \delta\omega^\lambda{}_\sigma x^\sigma\partial_\lambda\partial^\mu\phi(x) + \delta\omega^\mu{}_\rho\partial^\rho\phi(x)$. Thus

$$\begin{aligned}\frac{i\delta\omega^{\kappa\nu}}{2}[\partial^\mu\phi, M_{\kappa\nu}] &= -\delta\omega^\lambda{}_\sigma x^\sigma\partial_\lambda\partial^\mu\phi(x) + \delta\omega^\mu{}_\rho\partial^\rho\phi(x) \\ &= \frac{i\delta\omega^{\lambda\sigma}}{2}\mathcal{L}_{\lambda\sigma}\partial^\mu\phi(x) + \frac{i\delta\omega^{\lambda\sigma}}{2}(S_{V\lambda\sigma})^\mu{}_\rho\partial^\rho\phi(x)\end{aligned}$$

and result follows by comparing coefficients of $\delta\omega$.

b) Following steps to the result of 2.8(d), leads to

$$[\partial^\kappa \phi, [M^{\mu\nu}, M^{\rho\sigma}]] = \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\partial^\kappa\phi - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\partial^\kappa\phi + [S_V^{\mu\nu}, S_V^{\rho\sigma}]^\kappa{}_\eta \partial^\eta \phi$$

Plugging in commutator of M's show that commutator of S_V's must match.

c) The 12 block of $(-iS_V^{12})^n$ is $\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}^n$ which is $(-)^{n/2}I$ if n is even and $(-)^{(n-1)/2}\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$ if n is odd. Then in the 12 subblock:

$$\left[\sum_{n=0}^{\infty} \frac{1}{n!}(-i\theta S_V^{12})^n\right]_{12\text{Block}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

d) The 03 block of $(iS_V^{30})^n$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n$ which is I if n is even and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if n is odd. Then in the 03 subblock:

$$\left[\sum_{n=0}^{\infty} \frac{1}{n!}(i\eta S_V^{30})^n\right]_{03\text{Block}} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}$$

3. **The Scalar Field**

a) Show that the Klein-Gordon scalar wave equation is invariant under a Lorentz trasformation if the field ϕ transforms as a scalar field, *i.e.* $\phi'(x') = \phi(x)$. Here $x'^\mu = \Lambda^\mu_\nu x^\nu$ is a Lorentz transformation.

Solution: $\phi'(x') = \phi(\Lambda^{-1}x')$ so we have

$$\partial'_\mu \partial'_\nu \phi' = (\Lambda^{-1})_\mu{}^\rho (\Lambda^{-1})_\nu{}^\sigma \partial_\rho \partial_\sigma \phi$$

Since Λ^{-1} is a Lorentz transformation, $\eta^{\mu\nu}(\Lambda^{-1})_\mu{}^\rho(\Lambda^{-1})_\nu{}^\sigma = \eta^{\rho\sigma}$, so it follows that

$$\partial'^2\phi' = \eta^{\mu\nu}\partial'_\mu\partial'_\nu\phi' = \partial^2\phi' = m^2\phi'$$

as desired.

$b)$ Show by direct substitution of the continuum field expansions

$$\phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\hbar}{2\omega(\mathbf{k})}}(a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})$$
$$\pi(\mathbf{x}) = -i\int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\hbar\omega(\mathbf{k})}{2}}(a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})$$

into the formulas for the energy and momentum of the continuum scalar field that

$$H_\phi - E_0 = \int d^3k\hbar\omega(\mathbf{k})a^\dagger(\mathbf{k})a(\mathbf{k})$$
$$\mathbf{P} = -\int d^3x\pi(\mathbf{x})\nabla\phi(\mathbf{x}) = \int d^3k\hbar\mathbf{k}a^\dagger(\mathbf{k})a(\mathbf{k}).$$

Note that in the future we will be assuming units where $\hbar = 1$. I left $\hbar \neq 1$ in this problem to show the familiar Planck condition $E = \hbar\omega = h\nu$ and the de Broglie relation $\mathbf{P} = \hbar\mathbf{k}$. A difference between Srednicki's and my conventions is that I normalize creation and annihilation operators so that $[a, a^\dagger] = \delta$, compared to his $[a, a^\dagger] = 2\omega(\mathbf{k})(2\pi)^3\delta$. (see S (3.29)) This explains the apparent difference between the above equations and S (3.19).

Solution: When ϕ, Π are plugged into $H = \frac{1}{2}\int d^3x[\Pi^2 + (\nabla\phi)^2 + m^2\phi^2]$, the $\int d^3x$ produces delta functions:

$$\int d^3x\Pi^2 = \int d^3k\frac{\hbar\omega}{2}[a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k}) - a(\vec{k})a(-\vec{k}) - a^\dagger(\vec{k})a^\dagger(-\vec{k})]$$
$$\int d^3x(\nabla\phi)^2 = \int d^3k\frac{\hbar\vec{k}^2}{2\omega}[a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k}) + a(\vec{k})a(-\vec{k}) + a^\dagger(\vec{k})a^\dagger(-\vec{k})]$$
$$\int d^3xm^2\phi^2 = \int d^3k\frac{\hbar m^2}{2\omega}[a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k}) + a(\vec{k})a(-\vec{k}) + a^\dagger(\vec{k})a^\dagger(-\vec{k})]$$

Adding them up we find

$$H = \frac{1}{2}\int d^3x\hbar\omega[a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})] = E_0 + \int d^3x\hbar\omega a^\dagger(\vec{k})a(\vec{k})$$

Similar manipulations lead to

$$\vec{P} = -\int d^3x\Pi\nabla\phi = \int d^3k\frac{\hbar\vec{k}}{2}[a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})] = \int d^3k\hbar\vec{k}a^\dagger(\vec{k})a(\vec{k})$$

In this case the constant piece $\delta(\vec{0})\int d^3k\hbar\vec{k}/2 = 0$ upon integrating over directions.