Quantum Field Theory I

Problem Set 2

Due: 14 September 2007

1. S, Problem 3.2

   Solution: We first note $[H, a^\dagger(\mathbf{k}_i)] = \int \frac{d^3k}{(2\pi)^3 2\omega(\mathbf{k})} \omega(\mathbf{k}) a^\dagger(\mathbf{k})[a(\mathbf{k}), a^\dagger(\mathbf{k}_i)] = \omega(\mathbf{k}_i) a^\dagger(\mathbf{k}_i)$, by (3.29). Then

$$Ha_1^\dagger \cdots a_n^\dagger |0\rangle = \sum_{i=1}^n a_1^\dagger \cdots [H, a_i^\dagger] \cdots a_n^\dagger |0\rangle + a_1^\dagger \cdots a_n^\dagger H |0\rangle = \sum_{i=1}^n \omega_i a_1^\dagger \cdots a_n^\dagger |0\rangle + 0$$

2. S, Problem 3.3. The notation $\Lambda^{-1}\mathbf{k}$ means the spatial components of the four vector $(\Lambda^{-1})^\mu{}_\nu k^\nu$, in which we have set $k^0 = \sqrt{\mathbf{k}^2 + m^2}$. Note that $a, a^\dagger$ in Eq. (3.34) satisfy normalization (3.29). Work out how (3.34) would look with our normalization $[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = \delta^3(\mathbf{k} - \mathbf{k}')$. Take care that ω depends on $\mathbf{k}$.

   Solution: It follows from the Lorentz invariance of the scalar product $x \cdot k = \vec{x} \cdot \vec{k} - t\omega(\vec{k})$ that $(\Lambda^{-1}x) \cdot k = x \cdot (\Lambda k)$. We want to change d^3k integration variables to d^3k', where $k^{i\prime} = \Lambda^i{}_j k^j + \Lambda^i{}_0 \omega(\vec{k}) \equiv \Lambda \vec{k}$. As discussed in the text, $d^3k = d^3k'(\omega(\vec{k})/\omega(\vec{k}'))$. With our normalization of the a's, the integrand of the expression for $\phi(\Lambda^{-1}x)$ involves

$$\frac{d^3k}{\sqrt{\omega(\vec{k})}} a(\vec{k}) e^{ik\cdot(\Lambda^{-1}x)} = \frac{d^3k'}{\sqrt{\omega(\vec{k}')}} \sqrt{\frac{\omega(\Lambda^{-1}\vec{k}')}{\omega(\vec{k}')}} a(\Lambda^{-1}\vec{k}') e^{ik'\cdot x}$$

   Thus, comparing the a term of $U^{-1}\phi U$ to the right side with the prime removed from the dummy variable, we see

$$U(\Lambda)^{-1} a(\vec{k}) U(\Lambda) = \sqrt{\frac{\omega(\Lambda^{-1}\vec{k})}{\omega(\vec{k})}} a(\Lambda^{-1}\vec{k})$$

   Since Sredicki's $a_S = (2\pi)^{3/2}\sqrt{2\omega(\vec{k})} a(\vec{k})$, it follows that

$$U(\Lambda)^{-1} a_S(\vec{k}) U(\Lambda) = a_S(\Lambda^{-1}\vec{k})$$

   That $U(\Lambda)^{-1} a_S^\dagger(\vec{k}) U(\Lambda) = a_S^\dagger(\Lambda^{-1}\vec{k})$ follows immediately by conjugating both sides and using $U^\dagger = U^{-1}$. Using this formula but with $\Lambda \to \Lambda^{-1}$ shows that $U(\lambda) |k_1 \ldots k_n\rangle = |\Lambda k_1 \ldots \Lambda k_n\rangle$

3. We used the result that $U(t_2, t_1) = T \exp\left\{ -i \int\limits_{t_1}^{t_2} H(t) dt \right\}$ is unitary if $H(t)$ is self-adjoint.

a) Prove this by using the differential equation satisfied by $U(t, t_1)$.

Solution: $i\dot{U}(t, t_1) = H(t)U(t, t_1)$ implies $-i\dot{U}^\dagger = U^\dagger H^\dagger(t) = U^\dagger H(t)$ so $d(U^\dagger U)/dt = U^\dagger(H(t) - H(t))U = 0$. Then $U(t_1, t_1) = I$ implies $U^\dagger U = I$ for all t.

b) The result must also follow by expanding U and $U^\dagger$ in power series and collecting together all terms with N $H(t)$'s multiplied together. For each $N > 0$ these terms must cancel to give zero. Check this for $N = 1, 2$. [The ambitious can try to construct a proof for general N but do not waste time on this. It is simple if you hit on the right way, very nasty otherwise.]

Solution: The expansion of U involves time ordered products of H, whereas the expansion of $U^\dagger$ involves anti-time ordered products of H. When we expand $U^\dagger U$ and collect together all terms with N $H(t)$'s multiplied together, we find $N+1$ terms $H(t_N)H(t_{N-1}) \cdots H(t_1)$, each term with different limits on the ranges of the t_i as follows: $t_N < t_{N-1} < \cdots < t_2 < t_1$; $t_N < t_{N-1} < \cdots < t_2, t_1$; $t_N < t_{N-1} < \cdots < t_3, t_2 > t_1$; $t_N < t_{N-1} < \cdots < t_4, t_3 > t_2 > t_1$; ... $t_N > t_{N-1} > \cdots > t_2 > t_1$. Here the commas in each set of inequalities separate groups of t_i's that range independently of each other. The coefficient of each term is ± 1 where the sign alternates from one term to the next. Thus inspecting the first two ranges we see that the first range removes the part of the second range where $t_2 < t_1$ and converts it to the range $t_N < t_{N-1} < \cdots < t_3 < t_2 > t_1$. Combining this range with the third range, the latter becomes $t_N < t_{N-1} < \cdots < t_3 > t_2 > t_1$. This process continues until the next to the last range, $t_N, t_{N-1} > \cdots > t_2 > t_1$, is converted to $t_N > t_{N-1} > \cdots > t_2 > t_1$, which then cancels the last range. Explicitly for $N = 2$, the three ranges are $t_2 < t_1$; t_2, t_1, and $t_2 > t_1$. Combining the first two ranges gives $t_2 > t_1$ which cancels the third range.

4. The familiar form of the adiabatic theorem in the Schrödinger picture says that if $H_S(t)$ is the Hamiltonian, with eigenstates and eigenvalues $H_S(t)\,|n, t\rangle = E_n(t)\,|n, t\rangle$, and we solve the Schrödinger equation $i\frac{d}{dt}\,|\psi(t)\rangle = H_S(t)\,|\psi(t)\rangle$ with $|\psi(0)\rangle = |n, 0\rangle$, then under adiabatic conditions $|\psi(t)\rangle \approx e^{i\phi(t)}\,|n, t\rangle$ *i.e.* even though $H_S(t)$ is very different from $H_S(0)$, provided the change is slow enough, eigenstates move into eigenstates. The translation of this result into the Hiesenberg picture says that if $H(t)$ is the Heisenberg Hamiltonian derived from an adiabatically changing $H_S(t)$ then the eigenstates of $H(t)$ are approximately the same as the eigenstates of $H(0)$.

a) Suppose that $U(t)$ is the solution of $idU(t)/dt = H_S(t)U(t)$ with $U(0) = I$. Express the Schrödinger picture adiabatic theorem in terms of $U(t)$, write down the Heisenberg Hamiltonian in terms of $H_S(t)$ and $U(t)$, and hence establish the translation stated above.

Solution:Let $U(t)$ be the solution of $idU(t)/dt = H_S(t)U(t)$ with $U(0) = I$. Then $|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$. The Heisenberg picture Hamiltonian is $H(t) = U^\dagger(t)H_S(t)U(t)$, so that the eigenstates of $H(t)$ are $U^\dagger(t)\,|n,t\rangle$. The adiabatic theorem states that $|\psi(t)\rangle \approx |n,t\rangle$ if $|\psi(0)\rangle = |n,0\rangle$. This means that $|\psi(0)\rangle \approx U^\dagger\,|n,t\rangle$, which is an eigenstate of $H(t)$.

b) Consider the following simple case. The system is a harmonic oscillator with an external force. Using $a = (p - i\omega q)/\sqrt{2\omega}$, we can take

$$H(t) = \omega a^\dagger(t)a(t) - F(t)a^\dagger(t) - F^*(t)a(t)$$

where all operators are Heisenberg picture operators and $F(t)$ is a slowly changing *c*-number, with $F(0) = 0$. Write down the Heisenberg equation of motion for $a(t)$. Solve it for $a(T)$ in terms of $a(0)$ and the values of $F(t)$ in the range $0 < t < T$. [$a(t)$ is an operator but you will be able to solve just as if you were solving an ordinary first order inhomogeneous linear differential equation for a c-number function of t. Why?] Show that

$$a(T) - \frac{1}{\omega}F(T) = e^{-i\omega T}\left\{a(0) - \int_0^T \frac{e^{i\omega t}}{\omega}\frac{dF}{dt}dt\right\}$$

and hence verify the Heisenberg picture adiabatic theorem.

Solution: $i\dot{a}(t) = [a(t), H(t)] = \omega a(t) - F(t)$ is equivalent to $i\frac{d}{dt}\left(a(t)e^{i\omega t}\right) = -e^{i\omega t}F(t)$, which can be directly integrated:

$$a(t)e^{i\omega t} - a(0) = i\int_0^t dt' F(t')e^{i\omega t'} = \frac{1}{\omega}\int_0^t dt' F(t')\frac{d}{dt'}e^{i\omega t'} = \frac{F(t)e^{i\omega t}}{\omega} - \int_0^t dt' \frac{e^{i\omega t'}}{\omega}\frac{dF(t')}{dt'}.$$

Simple substitution establishes

$$a(T) - \frac{1}{\omega}F(T) = e^{-i\omega T}\left\{a(0) - \int_0^T \frac{e^{i\omega t}}{\omega}\frac{dF}{dt}dt\right\}$$

Adiabatic conditions then imply $a(T) - F(T)/\omega \approx e^{-i\omega T}a(0)$. Note that the Hamiltonian can be written

$$H(t) = \omega(a(t) - F(t)/\omega)^\dagger(a(t) - F(t)/\omega) - |F(t)|^2/\omega$$

which shows that $(a(T) - F(T)/\omega)$ is the lowering operator that determines the eigenstates of $H(T)$. Since it is approximately $a(0)$, we see that the initial eigenstates remain eigenstates.