Quantum Field Theory I

Problem Set 3

Due: 28 September 2007

1. The Complex Scalar Field. We can write the free complex (non-hermitian) scalar field as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{p})}} \left(a(\mathbf{p})e^{ip\cdot x} + b^\dagger(\mathbf{p})e^{-ip\cdot x}\right), \tag{1}$$

where $a^\dagger(\mathbf{p})$ creates a spin 0 particle and $b^\dagger(\mathbf{p})$ creates the associated antiparticle. Their commutation relations are

$$[a(\mathbf{p}), a^\dagger(\mathbf{p'})] = [b(\mathbf{p}), b^\dagger(\mathbf{p'})] = \delta(\mathbf{p'} - \mathbf{p}),$$

with all other commutators vanishing.

(*a*) Show from (1) that ϕ and $\dot{\phi}^\dagger$ satisfy the equal time commutation relations

$$[\phi(\mathbf{x}, t), \dot{\phi}^\dagger(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}),$$

and that ϕ satisfies the Klein-Gordon Equation

$$(\nabla^2 - \frac{\partial^2}{\partial t^2} - m^2)\phi(x) = 0.$$

Solution

$$\begin{aligned}
[\phi(\vec{x}, t), \dot{\phi}^\dagger(\vec{y}, t)] =& \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega}} \int \frac{d^3p'(-i\omega')}{(2\pi)^{3/2}\sqrt{2\omega'}} [a(\vec{p})e^{ix\cdot p} + b^\dagger(\vec{p})e^{-ix\cdot p}, b(\vec{p}')e^{iy\cdot p'} - a^\dagger(\vec{p}')e^{-iy\cdot p'}] \\
=& \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega}\sqrt{2\omega'}} d^3p'(-i\omega')\delta(\vec{p} - \vec{p}') \left(-e^{i(\vec{x}-\vec{y})\cdot\vec{p}} - e^{-i(\vec{x}-\vec{y})\cdot\vec{p}}\right) \\
=& i \int \frac{d^3p}{2(2\pi)^3} \left(e^{i(\vec{x}-\vec{y})\cdot\vec{p}} + e^{-i(\vec{x}-\vec{y})\cdot\vec{p}}\right) = i\delta(\vec{x} - \vec{y})
\end{aligned}$$

$$(\nabla^2 - \frac{\partial^2}{\partial t^2} - m^2)\phi = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(\vec{p})}} (-m^2 - p^2) \left(a(\vec{p})e^{ip\cdot x} + b^\dagger(\vec{p})e^{-ip\cdot x}\right) = 0$$

because $p^0 = \sqrt{m^2 + \vec{p}^2}$.

(*b*) Alternatively we can work with hermitian fields ϕ_1, ϕ_2 defined by $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. Show that

$$[\phi_k(\mathbf{x}, t), \dot{\phi}_l(\mathbf{y}, t)] = i\delta_{kl}\delta^3(\mathbf{x} - \mathbf{y}).$$

Thus $\dot{\phi}_k \equiv \pi_k$ is the conjugate momentum to ϕ_k, and the Hamiltonian is just the sum two commuting terms, one for each of the hermitian fields ϕ_1, ϕ_2:

$$H = \int d^3x \frac{1}{2} \sum_k (\pi_k^2 + (\nabla\phi_k)^2 + m^2\phi_k^2).$$

Solution: Solve $\phi_1 = (\phi + \phi^\dagger)/\sqrt{2}$ and $\phi_2 = -i(\phi - \phi^\dagger)/\sqrt{2}$. Then

$$[\phi_1, \dot{\phi}_2] = i[\phi, \dot{\phi}^\dagger]/2 - i[\phi^\dagger, \dot{\phi}]/2 = -\delta/2 + \delta/2 = 0.$$

Similarly, $[\phi_2, \dot{\phi}_1] = 0$.

$$[\phi_1, \dot{\phi}_1] = [\phi, \dot{\phi}^\dagger]/2 + [\phi^\dagger, \dot{\phi}]/2 = i\delta/2 + i\delta/2 = i\delta$$

(*c*) Now returning to the original nonhermitian field ϕ, and defining $\pi = (\dot{\phi}_1^\dagger - i\dot{\phi}_2^\dagger)/\sqrt{2}$, show that

$$H = \int d^3x (\pi\pi^\dagger + \nabla\phi^\dagger\nabla\phi + m^2\phi^\dagger\phi),$$

and the equal time commutation relations become

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}). \tag{2}$$

Solution: First note:

$$\pi\pi^\dagger = \frac{1}{2}(\dot{\phi}_1^2 + \dot{\phi}_2^2) = \frac{1}{2}\sum_k \pi_k^2$$

$$\phi^\dagger\phi = \frac{1}{2}\sum_k \phi_k^2$$

$$\nabla\phi^\dagger \cdot \nabla\phi = \frac{1}{2}\sum_k \nabla\phi_k \cdot \nabla\phi_k$$

Then comparing with the Cartesian form for H we see

$$H = \int d^3x (\pi\pi^\dagger + \nabla\phi^\dagger\nabla\phi + m^2\phi^\dagger\phi)$$

and the equal time commutation relations become

$$[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] = i\delta^3(\mathbf{x}-\mathbf{y}). \tag{2}$$

(d) The minimal substitution rule for coupling an external electromagnetic field in a gauge invariant way is $\partial \to \partial - iqA$ in the Klein-Gordon equation. (Note that the appearance of i in this rule is what dictates that electromagnetism must couple to a complex (*i.e.* nonhermitian field).) Show that the resulting equation follows from the Heisenberg equations derived from the Hamiltonian

$$H = \int d^3x(\pi\pi^\dagger + (\nabla + iq\mathbf{A})\phi^\dagger(\nabla - iq\mathbf{A})\phi + m^2\phi^\dagger\phi + iqA_0(\pi\phi - \phi^\dagger\pi^\dagger)),$$

and the commutation relations (2).

Solution

$$\begin{aligned} i\dot{\phi} = [\phi, H] = i(\pi^\dagger + iqA_0)\phi \quad &\to \quad \pi^\dagger = \dot{\phi} - iqA_0\phi \\ i\dot{\pi}^\dagger = [\pi^\dagger, H] = i(\nabla - iq\mathbf{A})^2\phi - im^2\phi + i^2qA_0\pi^\dagger) \quad &\to \quad \dot{\pi}^\dagger - iqA_0\pi^\dagger = (\nabla - iq\mathbf{A})^2\phi - m^2\phi \\ (\partial_0 - iqA_0)^2\phi = (\nabla - iq\mathbf{A})^2\phi - m^2\phi \quad &\to \quad (\partial - iqA)^2\phi - m^2\phi = 0 \end{aligned}$$

(e) We shall see later in the course that gauge invariant time evolution implies that a current defined in terms of the change in the Schrödinger picture Hamiltonian, under a small change in the potentials with canonical variables fixed (i.e. ϕ, its *spatial* derivities and Π are held fixed).,

$$U^\dagger(t)\delta H_S U(t) = -\int d^3x j_\mu(\mathbf{x},t)\delta A^\mu(\mathbf{x},t)$$

is conserved. From the Hamiltonian in part (d) use this principle and the Heisenberg equations to obtain the expression for the current

$$j_\mu(x) = -iq(\phi^\dagger\partial_\mu\phi - (\partial_\mu\phi^\dagger)\phi) - 2q^2A_\mu\phi^\dagger\phi.$$

Confirm that $\partial_\mu j^\mu = 0$ as a consequence of the Klein-Gordon equation coupled to A_μ.

Solution:

$$\delta H = \int d^3x iq\delta A^i[\phi^\dagger\nabla_i\phi - (\nabla_i\phi^\dagger)\phi - 2iqA_i\phi^\dagger\phi] - \int d^3x iq\delta A^0(\pi\phi - \pi^\dagger\phi^\dagger)$$

This implies

$$j_i = -iq(\phi^\dagger\partial_i\phi - (\partial_\mu\phi^\dagger)\phi) - 2q^2A_i\phi^\dagger\phi$$
$$j_0 = iq\delta(\pi\phi - \pi^\dagger\phi^\dagger) = iq(\dot{\phi}^\dagger + iqA_0\phi^\dagger)\phi - iq(\dot{\phi} - iqA_0\phi)\phi^\dagger = -iq(\phi^\dagger\dot{\phi} - \dot{\phi}^\dagger\phi) - 2q^2A_0\phi^\dagger\phi)$$

which are just the space and time components of $j_\mu = -iq(\phi^\dagger\partial_\mu\phi - (\partial_\mu\phi^\dagger)\phi) - 2q^2A_\mu\phi^\dagger\phi$. to confirm that $\partial_\mu j^\mu = 0$ we simply calculate $\partial_\mu j^\mu = -iq(\phi^\dagger\partial^2\phi - (\partial^2\phi^\dagger)\phi) - 2q^2\partial_\mu(A_\mu\phi^\dagger\phi)$. From KG eq, $\partial^2\phi = 2iqA^\mu\partial_\mu\phi + iq\phi\partial\cdot A + (m^2 + q^2A^2\phi$. Then $(\phi^\dagger\partial^2\phi - (\partial^2\phi^\dagger)\phi) = 2iqA^\mu\partial_\mu(\phi^\dagger\phi) + 2iq\phi^\dagger\phi\partial\cdot A = 2iq\partial_\mu(A^\mu\phi^\dagger\phi)$. This shows $\partial_\mu j^\mu = 0$.

(*f*) Work out the charge $Q = \int d^3x j^0$ in terms of creation and annihilation operators for the case of zero external field ($A_\mu = 0$).

Solution:

$$Q = \int d^3x j^0 = iq\int d^3x \frac{d^3p d^3p'}{(2\pi)^3\sqrt{4\omega\omega'}}\Big[(a^\dagger(\vec{p})e^{-ix\cdot p} + b(\vec{p})e^{ix\cdot p})(-i\omega')(a(\vec{p}')e^{ix\cdot p'} - b^\dagger(\vec{p}')e^{-ix\cdot p'})$$
$$- (i\omega)(a^\dagger(\vec{p})e^{-ix\cdot p} - b(\vec{p})e^{ix\cdot p})(a(\vec{p}')e^{ix\cdot p'} + b^\dagger(\vec{p}')e^{-ix\cdot p'})\Big]$$
$$= iq\int\frac{d^3p}{2\omega}\Big[(a^\dagger(\vec{p})a(\vec{p}) - b(\vec{p})b^\dagger(\vec{p}) - a^\dagger(\vec{p})b^\dagger(-\vec{p})e^{2i\omega t} + b(\vec{p})a^\dagger(-\vec{p})e^{-2i\omega t})(-i\omega)$$
$$- (i\omega)(a^\dagger(\vec{p})a(\vec{p} - b(\vec{p})b^\dagger(\vec{p}) + a^\dagger(\vec{p})b^\dagger(-\vec{p})e^{2i\omega t} - b(\vec{p})a^\dagger(-\vec{p})e^{-2i\omega t})\Big]$$
$$= q\int d^3p[a^\dagger(\vec{p})a(\vec{p}) - b(\vec{p})b^\dagger(\vec{p})] = q\int d^3p[a^\dagger(\vec{p})a(\vec{p}) - b^\dagger(\vec{p})b(\vec{p})] - \text{an infinite constant}$$

2. Using the Hamiltonian of the first problem, calculate the scattering ampitude for a single charged scalar particle to first order in a static potential potential $A_\mu(\vec{x})$. Specialize to the Coulomb potential $\vec{A} = 0$, $A^0 = e/(4\pi|\vec{x}|)$, and calculate the differential cross section for this process as a function of the initial particle energy and scatterring angle.

Solution: The interaction part of the hamiltonian is in interaction picture

$$H' = \int d^3x[iqA^\mu(\phi_I^\dagger\nabla_\mu\phi_I - \nabla_\mu\phi_I^\dagger\phi_I) + q^2\vec{A}^2\phi_I^\dagger\phi_I]$$

Here we used $\pi_I = \dot{\phi}_I^\dagger$ because interaction picture fields satisfy the zero field equations of motion. Note that the term quadratic in A involves spatial components only, but is second order in the field and so will not contribute to the first order amplitude. According to the Dyson formula the scattering amplitude is to first order

$$(-i)\int d^4x\,\langle 0|\, a_I(\vec{p}')iqA^\mu(\phi_I^\dagger\nabla_\mu\phi_I - \nabla_\mu\phi_I^\dagger\phi_I)a_I^\dagger(\vec{p})\,|0\rangle = \frac{q\tilde{A}_\mu(p-p')i(p+p')^\mu}{(2\pi)^3\sqrt{4\omega\omega'}}.$$

Here $\tilde{A}_\mu(k) = \int d^4x e^{ik\cdot x}A_\mu(x) = 2\pi\delta(k^0)\int d^3x e^{i\vec{k}\cdot\vec{x}}A_\mu(\vec{x}) \equiv 2\pi\delta(k^0)\tilde{A}_\mu(\vec{k})$ for static fields. Thus the Feynman amplitude for this process is

$$\mathcal{M} = q\tilde{A}_\mu(\vec{p}-\vec{p}')i(p+p')^\mu \rightarrow -iq(\omega+\omega')\tilde{A}^0(\vec{p}-\vec{p}') = -2iq\omega\tilde{A}^0(\vec{p}-\vec{p}')$$

for the Coulomb potential using energy conservation.

$$\tilde{A}^0(\vec{k}) = \int d^3x\frac{Q}{4\pi|\vec{x}|}e^{i\vec{k}\cdot\vec{x}} = Q\frac{2\pi}{4\pi}\int_0^\infty r\,dr\frac{2i\sin(kr)}{ikr} = \frac{Q}{\vec{k}^2}$$

Putting everything together we have

$$d\sigma = \frac{d^3p'}{(2\pi)^3 2\omega'}\frac{1}{2\omega v}2\pi\delta(\omega'-\omega)4\omega^2\frac{q^2Q^2}{(\vec{p}-\vec{p}')^4} = d\Omega\frac{q^2Q^2\omega^2}{4\pi^2(\vec{p}-\vec{p}')^4}$$

Now $(\vec{p}-\vec{p}')^2 = 2\vec{p}^2(1-\cos\theta) = 4\vec{p}^2\sin^2(\theta/2)$ so finally

$$\frac{d\sigma}{d\Omega} = \left(\frac{qQ}{8\pi}\right)^2\frac{m^2+\vec{p}^2}{\vec{p}^4\sin^4(\theta/2)}$$

3. Show without using perturbation theory that

$$\Delta_A(x,y) \equiv \frac{\langle out|\,T[\phi(x)\phi^\dagger(y)]\,|in\rangle}{\langle out|in\rangle}$$

is a Green's function for the differential operator $m^2 - (\partial - iqA)^2$:

$$[m^2 - (\partial - iqA)^2]\Delta_A(x,y) = -i\delta(x-y).$$

Solution: Because $[m^2 - (\partial - iqA)^2]\phi = 0$, the only non-zero contributions to $[m^2 - (\partial - iqA)^2]\Delta_A(x,y)$ come from time derivatives acting on the θ functions implicit in the time ordering symbol. the first time derivative gives

$$\partial_0 \Delta_A(x,y) = \delta(x^0 - y^0)\frac{\langle out|\,[\phi(x),\phi^\dagger(y)]\,|in\rangle}{\langle out|in\rangle} + \frac{\langle out|\,T[\dot\phi(x)\phi^\dagger(y)]\,|in\rangle}{\langle out|in\rangle}$$
$$= \frac{\langle out|\,T[\dot\phi(x)\phi^\dagger(y)]\,|in\rangle}{\langle out|in\rangle}$$

because ϕ and $\phi^\dagger$ commute at equal times. Then

$$\partial_0^2 \Delta_A(x,y) = \delta(x^0 - y^0)\frac{\langle out|\,[\dot\phi(x),\phi^\dagger(y)]\,|in\rangle}{\langle out|in\rangle} + \frac{\langle out|\,T[\ddot\phi(x)\phi^\dagger(y)]\,|in\rangle}{\langle out|in\rangle}$$
$$= -i\delta(x-y) + \frac{\langle out|\,T[\ddot\phi(x)\phi^\dagger(y)]\,|in\rangle}{\langle out|in\rangle}$$

by the canonical commutation relations $[\phi^\dagger, \pi^\dagger] = [\dot\phi - iqA_0\phi, \phi^\dagger] = [\dot\phi, \phi^\dagger] = -i\delta$. It follows that

$$[m^2 - (\partial - iqA)^2]\Delta_A(x,y) = -i\delta(x-y).$$

4. The general Wick's theorem follows very simply from the special case of products of a single pair of boson operators $a, a^\dagger$. Check that it gives the correct answer for $\langle 0|a^n a^{\dagger n}|0\rangle$.

Solution: The Wick expansion prescribes a term for each distinct pairing of the $2n$ operators in the matrix element which is the product of all the 2 point functions for each of the pairs. The possible two point functions are $\langle 0|aa^\dagger|0\rangle = 1$, $\langle 0|aa|0\rangle = 0$, and $\langle 0|a^\dagger a^\dagger|0\rangle = 0$. Thus the only nonzero terms will be those with every a paired with an $a^\dagger$. There are $n!$ such pairings so the answer is $n!$, the correct answer.