Quantum Field Theory I

Problem Set 4

Due: 8 October 2006

1. S, Problem 11.1

a) The vertex for the transition $A \to B + B$ is $-2ig$, so the lowest order Feynman amplitude is $\mathcal{M} = -2ig$. Then

$$d\Gamma = \frac{1}{2\omega_A}\frac{1}{4\pi^2}\frac{d^3p_1'}{4\omega_B(\vec{p}_1')\omega_B(\vec{p}_1 - \vec{p}_1')}(4g^2)\delta(\omega_B(\vec{p}_1') + \omega_B(\vec{p}_1 - \vec{p}_1') - \omega_A)$$
$$\to \frac{g^2 p_1' d\Omega}{16\pi^2\omega_A\omega_B(p_1')} = \frac{g^2 p_1' d\Omega}{8\pi^2 m_A^2} = \frac{g^2\sqrt{m_A^2 - 4m_B^2}\,d\Omega}{16\pi^2 m_A^2}$$

in the rest frame of particle A. To get the total rate we integrate over all angles, yielding a factor of 4π and divide by the symmetry factor 2 to account for the identical particles in the final state:

$$\Gamma = \frac{g^2\sqrt{m_A^2 - 4m_B^2}}{8\pi m_A^2}$$

b) The vertex for $\phi \to \chi^- + \chi^+$ is $-ig = \mathcal{M}$ to lowest order. Thus the squared amplituce is one quarter that of a). The phase space integral is identical to that in a) with the appropriae change of names. Since the particles in the final state are distinguishable, there is no symmetry factor of 2. Thus

$$\Gamma = \frac{g^2\sqrt{m_\phi^2 - 4m_\chi^2}}{16\pi m_\phi^2}$$

2. S, Problem 11.2

a) $s = -(p+k)^2 = m_e^2 + 2m_e\omega$, and $u = -(p-k')^2 = m_e^2 - 2m_e\omega'$, where we used the fact that the electron momentum $p = (m_e, \vec{0})$ in the lab sysem.

b) $t = -(k-k')^2 = -2\omega\omega'(1-\cos\theta)$, but also $t = 2m_e^2 - s - u = 2m_e(\omega_{\prime} - \omega)$. Thus

$$\cos\theta = 1 - \frac{m_e(\omega - \omega')}{\omega\omega'}$$

It is interesting to also write this relation as

$$\omega' = \frac{m\omega}{m + \omega(1-\cos\theta)}$$

which shows the angular dependence of the final photon frequency.

c) Plugging into $|\mathcal{T}|^2$ the expressions for s, u yields,

$$
\begin{aligned}
|\mathcal{T}|^2 &= 32\pi^2\alpha^2 \left[\frac{m^2 + m\omega + \omega\omega'}{\omega^2} + \frac{m^2 - m\omega' + \omega\omega'}{\omega'^2} - \frac{2m^2 + m(\omega - \omega')}{\omega\omega'} \right] \\
&= 32\pi^2\alpha^2 \left[m^2 \left(\frac{1}{\omega} - \frac{1}{\omega'} \right)^2 + 2m \left(\frac{1}{\omega} - \frac{1}{\omega'} \right) + \frac{\omega}{\omega'} + \frac{\omega'}{\omega} \right] \\
&= 32\pi^2\alpha^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta \right]
\end{aligned}
$$

The velocity of the photon is 1, and the phase space integral over the electron momentum and final photon energy yields the factor

$$
\frac{\omega' d\Omega}{64\pi^2 m\omega(m + \omega(1 - \cos\theta))} = \frac{\omega'^2 d\Omega}{64\pi^2\omega^2 m^2}
$$

Multiplying these expressions together yields (11.51).

$$
\frac{d\sigma}{d\Omega} = \frac{\alpha^2\omega'^2}{2\omega^2 m^2} \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta \right]
$$

3. S, Problem 11.3

a) We just need to write out $(k_1 \cdot k_2')(k_1' \cdot k_3') = k_{1\mu} k'_{3\nu} k_2'^{\mu} k_1'^{\nu}$ to factor out the $k1', k_2'$ dependence of the integrand. Then we can factor the overall integrasl as shown in S(11.54).

b) The left side of S(11.55) is

$$
\int \frac{d^3k_1' d^3k_2'}{4\pi^2 4\omega_1'\omega_2'} k_2'^{\mu} k_1'^{\nu} \delta(k_1' + k_2' - k)
$$

which must transform as a second rank tensor under Lorentz transformations. Since it only depends of k, the tensor indices must be carried by $\eta_{\mu\nu}$ or $k^\mu k^\nu$. Thus

$$
\int \frac{d^3k_1' d^3k_2'}{4\pi^2 4\omega_1'\omega_2'} k_2'^{\mu} k_1'^{\nu} \delta(k_1' + k_2' - k) = A(k^2)k^2\eta^{\mu\nu} + B(k^2)k^\mu k^\nu
$$

by Lorentz covariance. By dimensional analysis, however it must be homogeneous of degree 2 in k since we are assuming massless neutrinos. Thus A, B must actually be constants.

c) Since it is invariant, we can evaluate the integral in the rest frame of k which is timelike. Then $\omega_1' = \omega_2' = k_1' = k^0/2$, and

$$\int dLIPS_2(k) = \int \frac{d\Omega k_1'^2 dk_1'}{4\pi^2 k^{02}} \delta(2\omega_1' - k^0) = \int \frac{d\Omega}{32\pi^2} = \frac{1}{8\pi}.$$

d Now $k_1' \cdot k_2' = (k_1 + k_2)^2/2 = k^2/2$ and $k \cdot k_1' = k \cdot k_2' = k_1' \cdot k_2' = k^2/2$. Doing the contractions gives 2 equations for A, B:

$$k^2(4A + B) = \frac{k^2}{16\pi}, \qquad k^4(A + B) = \frac{k^4}{32\pi}$$

which imply $A = 1/96\pi$, $B = 1/48\pi$.

e) With all final particles taken as massless, the maximum electron energy is $m/2$. The integrand involves $A(k_1 - k_3')^2 k_1 \cdot k_3' + B k_1 \cdot (k_1 - k_3') k_3' \cdot (k_1 - k_3')$. In the rest frame of the muon and in the aprroximation that the electron is massless, this quantity reduces to $m^2 E_3'(3m - 4E_3')/(96\pi)$. Then we get

$$d\Gamma = \frac{32 G_F^2}{m} \frac{k_3'^2 dk_3' d\Omega_3}{16\pi^3 E_3'} \frac{m^2 E_3'(3m - 4E_3')}{96\pi} \approx \frac{E_e^2 m(3m - 4E_e) G_F^2}{12\pi^3} dE_e \equiv \frac{d\Gamma}{dE_e} dE_e$$

since $E_e \equiv E_3' \approx k_3'$ for massless electrons.

f) The integrand is a polynomial and the upper limit is $m/2$, with the result

$$\Gamma = \frac{m^5 G_F^2}{192\pi^3} = \frac{1}{\tau}$$

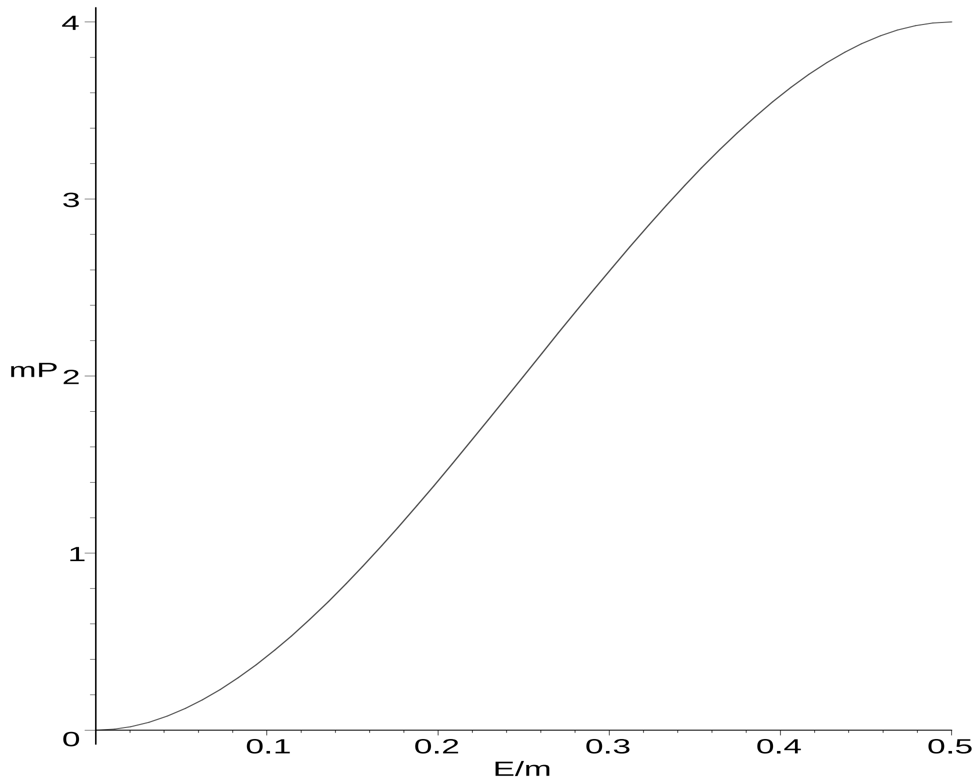
g) To convert the lifetime to GeV^{-1}, we divide by $\hbar = 6.5821 \times 10^{-25}$GeV-s. Then

$$G_F \approx \sqrt{\frac{192\pi^3 \cdot 6.5821}{1.0566^5 \cdot 2.197}} \cdot 10^{-7} \approx 1.164 \times 10^{-5}$$

compared with 1.166×10^{-5} in the particle data book.

h)

$$P(E_e) = \frac{16}{m}(E_e/m)^2(3 - 4E_e/m)$$



4. Path integral expression for the Partition function

The partition function $Z(\beta)$ at temperature β^{-1} is defined as $Tr\ e^{-\beta H} = \int dq \langle q|e^{-\beta H}|q\rangle$, where q is a complete set of coordinates. But we have an expression for $\langle q''|e^{-\beta H}|q'\rangle$ as an integral over all paths from q' at $t=0$ to q'' at $t=-i\beta$. So we get $Z(\beta) = \int Dq \exp\{i \int_0^\beta L(-i\tau)(-i)d\tau\}$ where the integral is over all paths with $q(-i\beta) = q(0)$. For the case of one degree of freedom with $L = \frac{1}{2}m\dot{q}^2 - V(q)$, this becomes

$$Z(\beta) = \int Dq \exp\left[-\int_0^\beta \left\{\frac{1}{2}m\left(\frac{dq}{d\tau}\right)^2 + V(q)\right\} d\tau\right],\ q(\beta) = q(0)$$

To define the integral, we break the interval 0 to β into N parts of length ϵ and replace $\int (dq/d\tau)^2\, d\tau$ by $(1/\epsilon)\sum_{r=1}^{N}(q_{r+1}-q_r)^2$ (with $q_{N+1} \equiv q_1$), $\int V(q)d\tau$ by $\epsilon\sum_{r=1}^{N} V(q_r)$, and then evaluate $(m/2\pi\epsilon)^{N/2}\int dq_1 \ldots dq_N$. In this exercise we shall calculate $Z(\beta)$ directly for the harmonic oscillator, $V(q) = \frac{1}{2}m\omega^2 q^2$.

a) Show that the formula for Z takes the form

$$Z(\beta) = \left(\frac{m}{2\pi\epsilon}\right)^{N/2} \int \prod_{r=1}^{N} dq_r \exp\left[-\frac{1}{2}\sum_{r,s=1}^{N} A_{rs}q_r q_s\right].$$

finding an explicit expression for the $N \times N$ matrix A_{rs}.

The discretized imaginary time action is

$$\frac{m}{2\epsilon}\sum_{r=1}^{N}(q_{r+1}-q_r)^2+\frac{m\omega^2\epsilon}{2}\sum_{r=1}^{N}q_r^2=\frac{1}{2}\sum_{r,s}q_r\left[\delta_{rs}\left(\frac{2m}{\epsilon}+m\omega^2\epsilon\right)-\frac{m}{\epsilon}(\delta_{r,s+1}+\delta_{r+1,s})\right]q_s$$

So

$$A_{rs}=\delta_{rs}\left(\frac{2m}{\epsilon}+m\omega^2\epsilon\right)-\frac{m}{\epsilon}(\delta_{r,s+1}+\delta_{r+1,s})$$

b) We can always change variables to linear combinations of q_r which makes A_{rs} diagonal. Show that $Z(\beta)=(m/\epsilon)^{N/2}\prod_{\nu=1}^{N}\lambda_\nu^{-1/2}$ where λ_ν are the eigenvalues of the equations $\sum_s A_{rs}q_s=\lambda q_r$. (In fact $\prod\lambda_\nu=\det A$.)

The transformation to normal coordinates is an orthogonal matrix so the Jacobian for the change of integration variables is unity. Thus

$$Z(\beta)=\left(\frac{m}{2\pi\epsilon}\right)^{N/2}\int\prod_{r=1}^{N}dq_r\exp\left[-\frac{1}{2}\sum_{\nu=0}^{N-1}\lambda_\nu q_r^2\right]=\left(\frac{m}{2\pi\epsilon}\right)^{N/2}\prod_\nu\left(\frac{2\pi}{\lambda_\nu}\right)^{1/2}=\left(\frac{m}{\epsilon}\right)^{N/2}\prod_\nu\lambda_\nu^{-1/2}.$$

c) Show that $\lambda_\nu=2(m/\epsilon)(1-\cos\theta_\nu)+m\epsilon\omega^2$, $\theta_\nu=2\nu\pi/N$, $\nu=0,\ldots,N-1$.

Making the ansatz that the eigenvectors of A_{rs} are of the form $v_r=Ne^{i\theta r}$, we easily read off the eigenvalue $\lambda(\theta)=\frac{2m}{\epsilon}(1-\cos\theta)+m\omega^2\epsilon$. The condition $v_1=v_{N+1}$ then implies $e^{iN\theta}=1$, which has solutions $\theta_\nu=2\pi\nu/N$, where ν is an integer. Restricting $\nu=0,1,\ldots,N-1$ gives N distinct eigenvalues, which proves that these eigenvectors are complete since A is an $N\times N$ symmetric matrix.

d) Evaluate $Z(\beta)$ using the identity $2(\cos N\theta-1)=\prod_{\nu=0}^{N-1}(2\cos\theta-2\cos\theta_\nu)$ (which is true because $\cos N\theta$ is a polynomial in $\cos\theta$ and the R.H.S. has the same zeros as the L.H.S. and the right coefficient of $\cos^N\theta$). Verify that it agrees with a direct evaluation of $\mathrm{Tr}\,e^{-\beta H}$ using standard raising and lowering operators $a, a^\dagger$.

We have $\prod_\nu\lambda^\nu=\left(\frac{m}{\epsilon}\right)^N\prod_\nu(2(1-\cos\theta_\nu)+\omega^2\epsilon^2)$ from which we infer that θ must satisfy $\cos\theta=1+\omega^2\epsilon^2/2$. For $\epsilon\to 0$, we learn that $-\theta^2\to\omega^2\epsilon^2$ or $\theta\to i\omega\epsilon$ so $N\theta\to i\omega\beta$. Thus

$$\prod_\nu\lambda^\nu=\left(\frac{m}{\epsilon}\right)^N 2(\cosh(\omega\beta)-1)=\left(\frac{m}{\epsilon}\right)^N 4(\sinh^2(\omega\beta/2)$$

From which it follows that $Z(\beta)=1/(2\sinh(\omega\beta/2))=e^{-\beta\omega/2}(1-e^{-\beta\omega})^{-1}$, the known answer. This is obtained by the formula $Z=\sum_n e^{-\beta E_n}=\sum_{n=0}^{\infty}e^{-\beta\omega(n+1/2)}$.