Quantum Field Theory I

Problem Set 5

Due: 17 October 2007

1. **Lorentz Covariance of Dirac Equation** Let $\Lambda^\mu{}_\nu$ represent a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. In the new frame, the Dirac equation is

$$\frac{1}{i}\gamma^\mu \partial'_\mu \psi'(x') + m\psi'(x') = 0$$

Note that the *same* gamma matrices are used in every Lorentz frame! Our goal is to express $\psi'(x')$ in terms of $\psi(x)$.

(a) Let $\sigma^{mn} = \frac{1}{2}i[\gamma^m, \gamma^n]$, where $m, n = 1, 2, 3$ are spatial indices. Show that $\sigma^{mn} = \epsilon_{mnk}\Sigma^k$ where $\mathbf{\Sigma}$ is the spin matrix for the Dirac equation.

(b) This result suggests that $M^{\mu\nu} = \frac{1}{4}i[\gamma^\mu, \gamma^\nu] \equiv \frac{1}{2}\sigma^{\mu\nu}$ is the generator of Lorentz transformations. Show that $M^{\mu\nu}$ satisfies the Lorentz algebra commutation relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\rho\nu} - \eta^{\nu\sigma}M^{\rho\mu}), \tag{1}$$

It will be helpful to first show that the commutators of M with γ are those with γ a four-vector, *i.e.*

$$[M^{\mu\nu}, \gamma^\rho] = i(\eta^{\mu\rho}\gamma^\nu - \eta^{\nu\rho}\gamma^\mu).$$

(c) Exploit these results to show that

$$\psi'(x') = e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\psi(\Lambda^{-1}x')$$

is a solution of the Dirac equation in the new frame provided $\psi(x)$ is a solution in the original frame and the matrix λ is related to Λ by

$$\Lambda^\mu{}_\nu = (e^{-\lambda})^\mu{}_\nu.$$

[Note that because of the group property of the $\sigma^{\mu\nu}$ established in the second part of this problem, it is sufficient to demonstrate this for infinitesimal λ.]

(d) Notice that $\sigma^{0k\dagger} = -\sigma^{0k}$, which means that the $\sigma^{\mu\nu}$ generate a non-unitary representation of the Lorentz group. Show that because of this the probability density $\psi^\dagger\psi$ transforms as the time component of a four vector rather than as a scalar field.

Solution:

a) $[\gamma^m, \gamma^n] = -\begin{pmatrix} [\sigma^m, \sigma^n] & 0 \\ 0 & [\sigma^m, \sigma^n] \end{pmatrix} = -2i\epsilon_{mnk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv -2i\epsilon_{mnk}\Sigma^k$

b) $[[\gamma^\mu, \gamma^\nu], \gamma^\rho] = \gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\nu\gamma^\mu = -4\gamma^\mu\eta^{\nu\rho} + 4\gamma^\nu\eta^{\mu\rho}$ by virtue of the Clifford algebra. This establishes $[M^{\mu\nu}, \gamma^\rho] = i(\gamma^\nu\eta^{\mu\rho} - \gamma^\mu\eta^{\nu\rho})$. Then $[M^{\mu\nu}, \gamma^\rho\gamma^\sigma]$ can be easily obtained verifying the Lorentz algebra.

c) $\partial'_\mu\gamma^\mu e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\psi(\Lambda^{-1}x') = e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}(\Lambda^{-1})^\alpha{}_\mu e^{i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\gamma^\mu e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\partial_\alpha\psi'$. The result follows if $e^{i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\gamma^\mu e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4} = (\Lambda^{-1})_\beta{}^\mu\gamma^\beta$. We can easily establish this last equality for infinitesimal transformations: $(\Lambda^{-1})^{\beta\mu} = \eta^{\beta\mu} - G^{\beta\mu}$: $e^{i\lambda_{\rho\tau}\sigma^{\rho\tau}/4}\gamma^\mu e^{-i\lambda_{\rho\tau}\sigma^{\rho\tau}/4} = \gamma^\mu - \frac{i}{4}[\gamma^\mu, \lambda_{\rho\tau}\sigma^{\rho\tau}] = \gamma^\mu + \frac{1}{2}(\gamma^\rho\lambda_\rho{}^\mu - \gamma^\rho\lambda^\mu{}_\sigma) = \gamma^\mu + \gamma^\rho\lambda_\rho{}^\mu$ where we used antisymmetry of λ. We get the desired result if $\lambda^{\beta\mu} = -G^{\beta\mu}$ which is the infinitesimal version of $\Lambda = e^{-\lambda}$. Note that $G^{\mu\nu}$ is antisymmetric in its indices.

d) $\psi^\dagger\psi \to \psi^\dagger e^{+i\lambda_{\mu\nu}\sigma^{\mu\nu\dagger}/4}e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\psi$. Inspection shows that $\sigma^{\mu\nu\dagger} = \gamma^0\sigma^{\mu\nu}\gamma^0$. ($\gamma^0$ anticommutes with σ^{0i} and commutes with σ^{ij}.) Thus $\psi^\dagger\psi \to \psi^\dagger\gamma^0 e^{+i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\gamma^0 e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\psi$. But $e^{+i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\gamma^\mu e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4} = (\Lambda^{-1})^\mu{}_\nu\gamma^\nu$ by c), establishing the result.

2. **Helicity basis for spin 1/2 particles.** Helicity is defined as the component of angular momentum of a particle along its momentum, *i.e.* $h = \mathbf{p}\cdot\mathbf{J}/|\mathbf{p}|$ with $\mathbf{J} = \mathbf{r}\times\mathbf{p} + \boldsymbol{\sigma}/2$. Note that since $\mathbf{p}\cdot(\mathbf{r}\times\mathbf{p}) = 0$, we may write, more simply, $h = \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}/2$, where $\hat{\mathbf{p}}$ is a unit vector parallel to $\mathbf{p}$.

(*a*) Show that the eigenvalues of h are $\pm 1/2$.

(*b*) Consider a single particle state $|\mathbf{p}, h\rangle$ with momentum $\mathbf{p}$ and helicity h. Show that rotations leave the helicity of the state unchanged.

(*c*) For $\mathbf{p} = p\hat{\mathbf{z}}$, $h = \sigma_z/2$ and we may take $|p\hat{\mathbf{z}}, h = \pm 1/2\rangle \equiv |p\hat{\mathbf{z}}, \sigma_z = \pm 1\rangle$ and we fix the ambiguity at $\mathbf{p} = 0$ by defining $|\mathbf{0}, h = \pm 1/2\rangle \equiv |\mathbf{0}, \sigma_z = \pm 1\rangle$. According to (b), we may define

$$|\mathbf{p}, h = \pm 1/2\rangle \equiv R_0(\mathbf{p})\,|p\hat{\mathbf{z}}, \sigma_z = \pm 1\rangle$$

where $R_0(\mathbf{p})$ is a standardized rotation that takes $p\hat{\mathbf{z}}$ into $\mathbf{p}$. Let θ, ϕ be the polar angles of $\mathbf{p}$. Then take

$$R_0(\mathbf{p}) \equiv e^{-i\phi J_z}e^{-i\theta J_y}e^{+i\phi J_z}.$$

With this definition and using spinor notation

$$|\mathbf{p}, \sigma_z = 1\rangle \equiv |\mathbf{p}\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\mathbf{p}, \sigma_z = -1\rangle \equiv |\mathbf{p}\rangle \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

show that

$$|\mathbf{p}, h = 1/2\rangle = |\mathbf{p}\rangle \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \equiv |\mathbf{p}\rangle\, \chi_{1/2}(\mathbf{p})$$

$$|\mathbf{p}, h = -1/2\rangle = |\mathbf{p}\rangle \begin{pmatrix} -e^{-i\phi}\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} \equiv |\mathbf{p}\rangle\, \chi_{-1/2}(\mathbf{p}).$$

(*d*) Using the results of (c) prove the following identities:

$$\begin{aligned} \chi_\lambda(\mathbf{p}) =& -ie^{i\lambda\pi + 2i\lambda\phi}\chi_{-\lambda}(-\mathbf{p}) \\ i\sigma^2\chi_\lambda(\mathbf{p}) =& e^{2i\lambda\phi}\chi_\lambda^*(-\mathbf{p}). \end{aligned}$$

Solution

a) $h = \vec{p}\cdot\vec{\sigma}/2|\vec{p}|$ satisfies $h^2 = 1/4$ so its eigenvalues are $\pm 1/2$.

b) Under rotations, generated by $\vec{J}$, both $\vec{p}$ and $\vec{\sigma}$ transform as vectors. Thus h, the scalar product of two vectors, commutes with rotations, implying that rotations do not change the helicity.

c) We evaluate

$$R_0(\vec{p})\,|p\hat{z}\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\vec{p}\rangle\, \chi_{1/2}(\vec{p})$$

where $\chi^{1/2}(\vec{p}) = e^{-i\phi\sigma^3/2}e^{-i\theta\sigma^2/2}e^{i\phi\sigma^z/2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= e^{i\phi/2}e^{-i\phi\sigma^3/2}(\cos\theta/2 - i\sigma^2\sin\theta/2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}$ Similarly we find $\chi_{-1/2}(\vec{p}) = e^{-i\phi\sigma^3/2}e^{-i\theta\sigma^2/2}e^{i\phi\sigma^z/2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= e^{-i\phi/2}e^{-i\phi\sigma^3/2}(\cos\theta/2 - i\sigma^2\sin\theta/2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -e^{-i\phi}\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}$

d) Putting $\theta_{-\vec{p}} = \pi - \theta_{\vec{p}}$ and $\phi_{-\vec{p}} = \pi + \phi_{\vec{p}}$, we have $\chi_{1/2}(-\vec{p}) = -e^{i\phi}\chi_{-1/2}(\vec{p})$ and $\chi_{-1/2}(-\vec{p}) = e^{-i\phi}\chi_{1/2}(\vec{p})$. We easily check that these two relations imply $\chi_\lambda(\vec{p}) = -ie^{i\pi\lambda + 2i\lambda\phi}\chi_{-\lambda}(-\vec{p})$. First evaluate $i\sigma_2\chi_\lambda(\vec{p}) = (-)^{\lambda+1/2}\chi^*_{-\lambda}(\vec{p})$. Then use the first identity to get the result.

3. Demonstrate that the Dirac bilinears, $\bar{\psi}(x)\psi(x)$, $\bar{\psi}(x)\gamma_5\psi(x)$, $\bar{\psi}(x)\gamma^\mu\psi(x)$, $\bar{\psi}(x)\gamma_5\gamma^\mu\psi(x)$, and $\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$, transform under Lorentz and parity transformations as scalar, pseudoscalar, vector, axial vector, and second rank tensor

fields respectively. (The terms pseudo- and axial refer to opposite than "normal" parity properties. For a normal 4-vector, the time component is even and the space component is odd under parity. A normal scalar is even under parity.)

Solution Because $\sigma^{\mu\nu\dagger} = \gamma^0\sigma^{\mu\nu}\gamma^0$, $\bar{\psi}A\psi \to \bar{\psi}e^{+i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}Ae^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\psi$ for any matrix A. If $A = I$ or $A = \gamma_5$ the bilinear is a scalar under proper Lorentz transformations since both commute with $\sigma^{\mu\nu}$. If $A = \gamma^\mu$ or $A = \gamma_5\gamma^\mu$, problem 1b) above shows that the bilinear is a four vector under proper L.T.'s, and $\gamma^\mu\gamma^\nu$ (and hence also $\sigma^{\mu\nu}$) is a two index tensor under proper L.T.'s. Since the parity transformation on ψ includes multiplication by $\beta = \gamma^0$, and $\gamma_5\gamma^0 = -\gamma^0\gamma_5$, an extra γ_5 in the bilinear produces an extra $-$ under parity converting a proper tensor to a pseudo-tensor.

4. Show that the Dirac equation implies that the free electron has a Landé g factor of 2. [Hint: Identify the magnetic moment by examining the Dirac equation for a weak slowly varying magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ in the limit that the electron is moving slowly. The field enters the Dirac equation via the substitution $\nabla \to \nabla - iq\mathbf{A}$. In this situation, after eliminating the lower two components of $\psi = \binom{\phi}{\chi}$ in favor of the upper two components, the D.E. should go over to the NR Schrodinger equation with a term that can be identified as $-\vec{\mu} \cdot \vec{B}$.]

Solution:

For a slow electron it is convenient to write $\psi = e^{-imt}\psi_0$. Then time derivatives of ψ_0 can be neglected compared to m and we find $\chi_0 \approx \vec{\sigma} \cdot (\nabla - iq\vec{A})\phi_0$. Then we find the equation for ϕ_0:

$$i\dot{\phi}_0 \approx -\frac{1}{2m}[\vec{\sigma} \cdot (\nabla - iq\vec{A})]^2\phi_0 = -\frac{1}{2m}(\nabla - iq\vec{A})^2\phi_0 - \frac{q}{2m}\vec{\sigma} \cdot \vec{B}\phi_0$$

where we used $\sigma^k\sigma^l = \delta_{kl} + i\epsilon^{klm}\sigma^m$. The last term shows that the spin part of the magnetic moment is $\vec{\mu} = q\vec{\sigma}/2m = q\vec{S}/m \equiv gq\vec{S}/2m$. That is $g = 2$.

5. The Dirac equation for an electron in the Coulomb field of a proton is

$$(i\gamma \cdot \partial + \gamma^0\frac{e^2}{4\pi r} - m)\psi = 0.$$

(*a*) Writing the wave function for a stationary state in the form

$$\psi(x) = e^{-imt-iEt}\begin{pmatrix}\phi(\mathbf{x}) \\ \chi(\mathbf{x})\end{pmatrix}$$

where ϕ and χ are 2 component spinors, solve for χ in terms of ϕ and show that ϕ satisfies the equation

$$\left\{-\boldsymbol{\sigma}\cdot\nabla\left(2m+E+\frac{e^2}{4\pi r}\right)^{-1}\boldsymbol{\sigma}\cdot\nabla-\frac{e^2}{4\pi r}\right\}\phi=E\phi.$$

Solution: Writing D.E. in std rep, gives

$$(m+E)\phi=\left(m-\frac{e^2}{4\pi r}\right)\phi+\frac{1}{i}\vec{\sigma}\cdot\nabla\chi;\quad (m+E)\chi=\left(-m-\frac{e^2}{4\pi r}\right)\chi+\frac{1}{i}\vec{\sigma}\cdot\nabla\phi$$

Solve 2nd eq for χ, plug in first to get desired eq.

(*b*) By taking suitable limits on the Dirac equation in this form, find the first relativistic corrections to the nonrelativistic Schrödinger equation. Identify the correction terms with the familiar spin-orbit and relativistic kinetic energy correction terms used to understand the fine structure splittings in junior quantum mechanics, paying attention to any differences. What can you conclude about the splittings of the 2s and 2p levels of hydrogen?

Solution: For zeroth order, $E, e^2/4\pi r=O(m\alpha^2)$ and $\nabla=O(m\alpha)$. Thus it is enough to keep just 2 terms in expansion $(2m+E+e^2/4\pi r)^{-1}\approx(1-E/2m-e^2/8\pi mr)/2m$, since the two ∇'s it multiplies are $O(m^2\alpha^2)$. For the same reason, we can replace $(E+e^2/4\pi r)\phi\to-(\nabla^2/2m)\phi$ in that term. To do this, we have to bring the $1/r$ next to the ϕ by using $[1/r,\vec{\sigma}\cdot\nabla]=\vec{r}\cdot\vec{\sigma}/r^3$. After all these changes the eq reads

$$\left\{-\frac{\nabla^2}{2m}-\frac{e^2}{4\pi r}\right\}\phi+\frac{1}{4m^2}\left\{\vec{\sigma}\cdot\nabla\frac{e^2\vec{r}\cdot\vec{\sigma}}{4\pi r^3}-\frac{\nabla^4}{2m}\right\}\phi=E\phi.$$

Next use $\sigma^k\sigma^l=\delta_{kl}+i\epsilon^{klm}\sigma^m$ to simplify the terms involving $\vec{\sigma}$

$$\left\{-\frac{\nabla^2}{2m}-\frac{e^2}{4\pi r}\right\}\phi+\frac{1}{4m^2}\left\{\frac{e^2\vec{r}}{4\pi r^3}\cdot\nabla+e^2\delta(\vec{r})+\frac{e^2\vec{\sigma}\cdot\vec{L}}{4\pi r^3}-\frac{\nabla^4}{2m}\right\}\phi=E\phi.$$

where $\vec{L}=-i\vec{r}\times\nabla$ is the orbital angular momentum. Finally we have to consider the normalization condition of the Dirac wavefunction

$$1=\int d^3x(\phi^\dagger\phi+\chi^\dagger\chi)\approx\int d^3x\phi^\dagger\left(1-\frac{\nabla^2}{4m^2}\right)\phi$$

which shows that the Schrodinger w.f. should be identified with $\Psi_S=\sqrt{1-\frac{\nabla^2}{4m^2}\phi}\approx\left(1-\frac{\nabla^2}{8m^2}\right)\phi$. To convert ϕ to Ψ_S in the equation we apply

$(1-\nabla^2/8m^2)$ to both sides. We pick up the commutator $[-\nabla^2, 1/r] = 4\pi\delta(\vec{r}) + 2(\vec{r}/r^3)\cdot\nabla$ in the first term. (The commutators from the second term are higher order than we keep).

$$\left\{-\frac{\nabla^2}{2m} - \frac{e^2}{4\pi r} + \frac{e^2}{8m^2}\delta(\vec{r}) + \frac{e^2\vec{\sigma}\cdot\vec{L}}{16m^2\pi r^3} - \frac{\nabla^4}{8m^3}\right\}\Psi_S = E\Psi_S.$$

The last three terms in the braces show the relativistic corrections. The fourth term is the standard spin orbit coupling, the fifth term is the standard relativistic correction to the kinetic energy, and the third term is the so-called Darwin term, which contributes only to $l = 0$ level shifts. Thus for $l \neq 0$ we read off the level shifts from any text on undergraduate quantum mechanics:

$$\Delta E_{njl} = -\frac{mc^2\alpha^4}{2n^3}\left[\frac{1}{j+1/2} - \frac{3}{4n}\right], \qquad l = j \pm 1/2 > 0$$

For $l = 0$ $(j = 1/2)$ the spin orbit term vanishes but we have the Darwin shift $\alpha\pi R_{n0}^2(0)/2m^2 = mc^2\alpha^4/2n^3$ added to the shift from the K.E. correction, and the result is the above for $j = 1/2$. Thus the D.E. equation predicts corrections that depend only on n and the total angular momentum j. In particular, the $2s_{1/2}$ and $2p_{1/2}$ levels remain degenerate. Experimentally they are not degenerate, split by the famous Lamb shift, which is caused by QED radiative corrections.