Quantum Field Theory I

Problem Set 6

Due: 29 October 2007

1. **Majorana Representation.**

   (*a*) Find a representation for the γ matrices such that they are all pure imaginary.

   Solution: There are of course many possibilities. One way is to simply rearrange the standard rep γ's we already have. For example $\gamma^0_{\text{Maj}} = i\gamma^1$, $\gamma^1_{\text{Maj}} = i\gamma^0$, $\gamma^2_{\text{Maj}} = \gamma^2$,$\gamma^3_{\text{Maj}} = i\gamma_5$ does the trick.

   (*b*) In this representation work out the charge conjugation transformation, *i.e.* find $C^{-1}\psi C$.

   Solution: With imaginary γ's the hermitian conjugate of the Dirac equation is also the D.E. Thus, we can take the charge conjugation transformation to be $C^{-1}\psi C = (\psi^\dagger)^T$.

   (*c*) Show that in D space-time dimensions the γ matrices must be at least of size $2^{D/2} \times 2^{D/2}$ for D even and at least $2^{(D-1)/2} \times 2^{(D-1)/2}$ for D odd.

   Solution: When D is even we can find $D/2$ commuting matrices by pairing them up: $\gamma^0\gamma^1, \gamma^2\gamma^3, ..., \gamma^{D-2}\gamma^{D-1}$. Since each of these commuting matrices has at least two distinct eigenvalues ± 1 the matrices have to be at least $2^{D/2}$ dimensional. If D is odd, $D-1$ is even so we can find $D-1$ gamma marices of dimension $2^{(D-1)/2}$. But then the product of all $D-1$ of these gamma matrices always gives one more so $2^{(D-1)/2}$ is the minimal matrix size in this case.

   (*d*) *Challenge* (for extra credit) See if you can work out in which dimensions it is possible to find a Majorana representation assuming matrices of the minimal size.

   Solution: When $D = 2m$, one can find a representation like the standard representation with $m+1$ real $2^m \times 2^m$ matrices, say, $\gamma^0, \gamma^1, \gamma^3, ..., \gamma^{D-1}$, and $m-1$ imaginary matrices, say, $\gamma^2, \gamma^4, ..., \gamma^{D-2}$. The matrix $C = \gamma^2\gamma^4 \cdots \gamma^{D-2}$ commutes with the real matrices and anticommutes with the imaginary ones if m is odd, and does the reverse if m is even. Thus $C^{-1}\gamma^{\mu *}C = (-)^{m-1}\gamma^\mu$. Write $\gamma^\mu_{\text{Maj}} = U^\dagger\gamma^\mu U$. The condition they are all imaginary is

$$U^\dagger\gamma^\mu U = -U^T\gamma^{\mu *}U^* = (-)^m U^T C\gamma^\mu C^{-1}U^*$$

   which can be rewritten $\gamma^\mu UU^TC = (-)^m UU^TC\gamma^\mu$. Thus $UU^TC \propto I$ if m is even and $UU^TC \propto \gamma_{D+1}$ if m is odd, where $\gamma_{D+1} \propto \gamma^0\gamma^1 \cdots \gamma^{D-1}$ is the generalization of γ_5. Since UU^T is a symmetric matrix, the condition for a Majorana representation is $C = C^T$ for m even and $C\gamma_{D+1} = (C\gamma_{D+1})^T$ for m odd. The γ's in C are imaginary and anti-hermitian, and so are symmetric. Calculating $C^T = \gamma^{D-2} \cdots \gamma^4 \gamma^2 = (-)^{\sum_{i=1}^{m-2} i} C = (-)^{(m-2)(m-1)/2} C$ so for m even, we require $(m-1)(m-2) = 4k$, or $m = 2 + 4k$, which translates to $D = 4 + 8k = 4, 12, 20, 28, \ldots$. When m is odd we need $(C\gamma_{D+1})^T = \gamma_{D+1}^T C^T = (-)^{(m-1)(m-2)/2} C\gamma_{D+1}^T$. Now $\gamma_{D+1}^T = \gamma_{D+1}$, so in this case the condition is $(m-1)/2 = 2k$ or $m = 1 + 4k$, which translates to $D = 2 + 8k = 2, 10, 18, 26, \ldots$ For $D = 2m + 1$ we can get $2m$ imaginary γ's if $m = 2 + 4k$ or $m = 1 + 4k$. We need one more antihermitian γ which we can take proportional to $\gamma^0_{\rm Maj}\gamma^1_{\rm Maj} \cdots \gamma^{D-2}_{\rm Maj}$. To make it antihermitian, we multiply it by $i^{D/2} = i^m$. This will be imaginary only for the $m = 1 + 4k$ series. Thus the possible odd dimension Majorana representations occur for $D = 3 + 8k = 3, 11, 19, 27, \ldots$.

2. How do the Dirac bilinears, $\bar\psi(x)\psi(x)$, $\bar\psi(x)\gamma_5\psi(x)$, $\bar\psi(x)\gamma^\mu\psi(x)$, $\bar\psi(x)\gamma_5\gamma^\mu\psi(x)$, and $\bar\psi(x)\sigma^{\mu\nu}\psi(x)$ transform under time reversal and charge conjugation when ψ is the second quantized field operator?

Solution: Under time reversal $\psi(\vec{x}, t) \to i\Sigma_2\psi(\vec{x}, -t)$, and any extra c-numbers are complex conjugated. Thus $\bar\psi A\psi \to \bar\psi(-i\Sigma_2)A^* i\Sigma_2\psi$, where $t \to -t$ is understood. So all we need do is work out $\Sigma_2 A^* \Sigma_2$ for all cases: $\Sigma_2 I^* \Sigma_2 = I$, $\Sigma_2(i\gamma_5)^*\Sigma_2 = -i\gamma_5$, $\Sigma_2\gamma^{0*}\Sigma_2 = \gamma^0$, $\Sigma_2\gamma^{k*}\Sigma_2 = -\gamma^k$ $\Sigma_2\gamma_5^*\gamma^{0*}\Sigma_2 = \gamma_5\gamma^0$, $\Sigma_2\gamma_5^*\gamma^{k*}\Sigma_2 = -\gamma_5\gamma^k$ $\Sigma_2\sigma^{0i*}\Sigma_2 = +\sigma^{0i}$, $\Sigma_2\sigma^{ij*}\Sigma_2 = -\sigma^{ij}$. In summary for the 5 basic bilinears the multiplicative sign associated with time reversal is,respectively, $(+, -, +, +, -)(-)^S$ where S in the number of *spatial* indices.

Charge conjugation is $\psi \to i\gamma^2(\psi^\dagger)^T$. Then $\bar\psi A\psi \to \psi^T i\gamma^2\gamma^0 A i\gamma^2(\psi^\dagger)^T = -\psi^\dagger i\gamma^2 A^T\gamma^0 i\gamma^2\psi = -\bar\psi\gamma^0 i\gamma^2 A^T\gamma^0 i\gamma^2\psi$. We find $-\gamma^0 i\gamma^2 I^T\gamma^0 i\gamma^2 = I$, $-\gamma^0 i\gamma^2(i\gamma_5)^T\gamma^0 i\gamma^2 = i\gamma_5$, $-\gamma^0 i\gamma^2(\gamma^\mu)^T\gamma^0 i\gamma^2 = -\gamma^\mu$, $-\gamma^0 i\gamma^2(\gamma_5\gamma^\mu)^T\gamma^0 i\gamma^2 = +\gamma_5\gamma^\mu$, $-\gamma^0 i\gamma^2(\sigma^{\mu\nu})^T\gamma^0 i\gamma^2 = -\sigma^{\mu\nu}$. In summary we have the signs $(+, +, -, +, -)$ for charge conjugation.

3. The results of problem 3 of set 5 are based on the fact that the similarity transformation

$$e^{i\lambda_{\mu\nu}\sigma^{\mu\nu}/4}\Gamma e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4} \tag{1}$$

with Γ any of the matrices $I, i\gamma_5, \gamma^\mu, \gamma_5\gamma^\mu, \sigma^{\mu\nu}$ performs a Lorentz transformation on the four-vector indices μ and ν. We have noted that the matrices

$$e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4} \tag{2}$$

give us the representation $D(1/2, 0) \oplus D(0, 1/2)$ of the Lorentz group.

(a) Show that the matrices

$$e^{i\lambda_{\mu\nu}\sigma^{T\mu\nu}/4}$$

are similar to the matrices (2). (A similarity transformation on matrix A is $S^{-1}AS$ for an invertible matrix S. Simply find an S that does the trick.)

Solution: It is enough to show that $\sigma^{T\mu\nu}$ is similar to$-\sigma^{\mu\nu}$. First note that $\gamma^{\mu *} = -i\gamma^2\gamma^\mu i\gamma^2$ and hence $\gamma^{T\mu} = -i\gamma^2\gamma^0\gamma^\mu\gamma^0 i\gamma^2$, so $\gamma^{\mu T}$ is similar to $-\gamma^\mu$. It follows that $\sigma^{T\mu\nu}$ is similar to $\sigma^{\nu\mu} = -\sigma^{\mu\nu}$.

(b) Because of (a), the transformation (1) may be viewed as belonging to the

$$(D(1/2,0) \oplus D(0,1/2)) \otimes (D(1/2,0) \oplus D(0,1/2)) \tag{3}$$

representation of the Lorentz group: Think of the matrix elements Γ_{ab} as a two-index bi-spinor, for which the Lorentz transformation reads

$$\Gamma'_{ab} = \Gamma_{cd}(e^{i\lambda_{\mu\nu}\sigma^{T\mu\nu}/4})_{ca}(e^{-i\lambda_{\mu\nu}\sigma^{\mu\nu}/4})_{db}$$

Find the decomposition of the tensor product representation (3) into irreducible representations of the Lorentz group.

Solution:

$$(D(1/2,0) \oplus D(0,1/2)) \otimes (D(1/2,0) \oplus D(0,1/2))$$

$$= (D(1/2,0) \otimes D(1/2,0)) \oplus (D(0,1/2) \otimes D(0,1/2))$$

$$\oplus (D(1/2,0) \otimes D(0,1/2)) \oplus (D(0,1/2) \otimes D(1/2,0))$$

$$= D(1,0) \oplus D(0,0) \oplus D(0,1) \oplus D(0,0) \oplus D(1/2,1/2) \oplus D(1/2,1/2)$$

(c) Relate the results of part (b) to the transformation properties of the matrices $I, i\gamma_5, \gamma^\mu, \gamma_5\gamma^\mu, \sigma^{\mu\nu}$.

Solution: $D(1/2,1/2)$ is the four vector representation of the Lorentz group and it occurs twice here: they must correspond to γ^μ and $\gamma_5\gamma^\mu$. The scalar representation $D(0,0)$ also occurs twice corresponding to $i\gamma_5$ and I. Finally $D(1,0) \oplus D(0,1)$ is a six dimensional representation which by the process of elimination must correspond to $\sigma^{\mu\nu}$, the antisymmetric rank two tensor representation. (note that the dimensions of these representations match.)

4. **Quantum Numbers of Positronium.**

a) A state of positronium (*i.e.* a hydrogenic e^+e^- atom) can be written as

$$|\Psi\rangle = \int d^3p \sum_{\mu_1\mu_2} F(\vec{p},\mu_1;-\vec{p},\mu_2) b^\dagger_{\vec{p}\mu_1} d^\dagger_{-\vec{p}\mu_2} |0\rangle \,.$$

where it is convenient to label spin not by helicity but by the spinor basis $\phi_\mu = \binom{1}{0}, \binom{0}{1}$ which are just eigenstates of σ^z, so that parity does not touch spin. When F corresponds to orbital angular momentum L and total spin $S(S = 0$ or $1)$ how does $|\Psi\rangle$ transform under parity and charge conjugation?

Solution: The positron and electron have negative relative intrinsic parity. Combining that with the parity $(-)^L$ associated with orbital angular momentum gives combined parity $(-)^{L+1}$. Thus the parity of S, P, D wave functions is $-,+,-$ respectively. Charge conjugation changes the electron and positron into each other. That is equivalent to exchanging the spin $(-)^{S+1}$ and orbital labels $(-)^L$ of the two particles plus an extra sign because $\{b^\dagger, d^\dagger\} = 0$: $C = (-)^{L+S}$. For triplet S, P, D this is $-,+,-$ and for singlet S, P, D it is $+,-,+$.

(*b*) It is a fact that an n photon state has charge conjugation eigenvalue $C = (-)^n$. If C is conserved in the spontaneous annihilation of positronium into photons, what is the minimum number of photons in the final state of the decay of the ground states of ortho-(S=1) and para-(S=0) positronium?

Solution: Since charge conjugation reverses the charge of a particle, we should have $Cj^\mu C^{-1} = -j^\mu$ (which you have checked in a previous problem). Thus $A_\mu \to -A_\mu$ under charge conjugation will make charge conjugation a symmetry of QED. This implies that a photon is odd under C and an n photon state has $C = (-)^n$. States of positronium with $L + S$ even have $C = +$ and hence annihilate only into an even number of photons. This includes the spin-0 ground state. States with $L+S$ odd, which include the spin-1 ground state must decay into an odd number of photons. One photon transitions must change C, so they can only occur between a state with $L + S$ even and a state with $L + S$ odd.