Quantum Field Theory I

Problem Set 7

Due: 14 November 2007

1. S, Problem 48.4, Solution

a) $\mathcal{M} = -ig\bar{u}_{s_1}(\mathbf{p})v_{s_2}(-\mathbf{p})$, so

$$\sum |\mathcal{M}|^2 = g^2\,\mathrm{Tr}(m-\gamma\cdot p)(-m-\gamma\cdot q) = 4g^2(-m^2-p\cdot q) = 2g^2(M^2-4m^2)$$

where we used $(p+q)^2 = -M^2$. The decay rate is then

$$\frac{2g^2(M^2-4m^2)}{2M}\int\frac{p\omega d\omega d\Omega}{16\omega^2\pi^2}\delta(M-2\omega) = \frac{g^2(M^2-4m^2)}{2M}\frac{p}{2M\pi} = \frac{g^2(M^2-4m^2)^{3/2}}{8M^2\pi}$$

b) Recall that $u(\vec{p}) = \sqrt{m+\omega}\binom{\phi}{\sigma\cdot p\phi/(m+\omega)}$ and $v(\vec{p}) = -\sqrt{m+\omega}\binom{\sigma\cdot pi\sigma^y\phi^*/(m+\omega)}{i\sigma^y\phi^*}$. Thus

$$\mathcal{M} = ig(m+\omega)\phi^\dagger_{s_1}\frac{-2\vec{\sigma}\cdot\vec{p}}{m+\omega}i\sigma^y\phi^*_{s_2} = 2g\phi^\dagger_{s_1}\vec{\sigma}\cdot\vec{p}\sigma^y\phi^*_{s_2}$$

If $\vec{p} = p\hat{z}$, $\vec{\sigma}\cdot\vec{p}\sigma^y = p\sigma^z\sigma^y = -ip\sigma^x$, so

$$|\mathcal{M}|^2 = 4p^2g^2|\phi^\dagger_{s_1}\sigma^x\phi^*_{s_2}|^2 = 4p^2g^2\delta_{s_1,s_2} = g^2(M^2-4m^2)$$

If $M = 2m$ the electron and positron have zero momentum and angular momentum, and therefore are in an odd parity state. Conservation of parity forbids the decay at threshold. Above threshold parity conservation requires that the pair have $l = odd$. Since the total angular momentum must be zero, the pair must have spin 1 and $l = 1$. If they have opposite spin $S^x = 0$, and then $L^x = 0$ by angular momentum conservation and the angular dependence will be $Y_{10} \propto \cos\theta$ where θ is the polar angle with respect to the x axis. This vanishes at $\theta = \pi/2$, i.e. if the momentum is in the yz-plane, which is the case considered.

c) For the helicity basis $\phi_s = \chi_s(\vec{p})$, and we have the identity $i\sigma^y\chi^*_s(\vec{p}) = e^{-2is\phi}\chi_s(-\vec{p})$. Then the squared amplitude reduces to

$$|\mathcal{M}|^2 = 4g^2|\chi^\dagger_{s_1}(\vec{p})\vec{\sigma}\cdot\vec{p}\chi_{s_2}(\vec{p})|^2 = 4p^2g^2\delta_{s_1,s_2} = g^2(M^2-4m^2)\delta_{s_1,s_2}.$$

Since $\vec{J}\cdot\vec{p} = \vec{S}\cdot\vec{p}$, and the initial state has zero total angular momentum, the spin along the momentum must add up to zero. But since the momenta are opposite for the two particles in the final state, opposite helicity means parallel spin, i.e. $S = \pm 1$ and is forbidden by angular momentum conservation.

d) This time

$$\sum |\mathcal{M}|^2 = g^2 \,\mathrm{Tr}\, i\gamma_5(m - \gamma \cdot p) i\gamma_5(-m - \gamma \cdot q) = -4g^2(-m^2 + p \cdot q) = 2g^2 M^2$$

This time the rate at threshold is non-zero because the spin zero initial state has odd parity as does e^+e^- at threshold. This explains why the rate is generally higher for this case.

e) This time

$$\mathcal{M} = -ig\bar{u}_{s_1}(\mathbf{p}) i\gamma_5 v_{s_2}(-\mathbf{p}) = g(m+\omega)\phi^\dagger_{s_1}\left(1 + \frac{\vec{p}^2}{(m+\omega)^2}\right) i\sigma^y \phi^*_{s_2} = 2g\omega\phi^\dagger_{s_1} i\sigma^y \phi^*_{s_2}$$

In the first case with $\vec{p}$ in the z direction and spin axis in the x direction, the σ^y flips the spin and so

$$|\mathcal{M}|^2 = 4g^2\omega^2\delta_{s_1,-s_2} = g^2M^2\delta_{s_1,-s_2}$$

By parity conservation $l = even$, and since $S_{max} = 1$ and the initial state has total angular momentum 0, we must then have $S = l = 0$. Since $S = 0$ the spins must be antialigned. In the second case, using the helicity spinor identities we see

$$|\mathcal{M}|^2 = g^2M^2|\chi^\dagger_{s_1}(\vec{p})\chi_{s_2}(\vec{p})|^2 = g^2M^2\delta_{s_1,s_2}$$

The argument that the spins are opposite still applies, but since momenta are also opposite, opposite spins mean like helicity.

2. Prove, without using perturbation theory, that the function

$$\frac{\langle out|\, T[\psi_\alpha(x)\bar{\psi}_\beta(y)]\, |in\rangle}{\langle out|in\rangle}$$

in the presence of an external electromagnetic field A_μ is a Green's function for the differential operator $[i\gamma \cdot \partial + q\gamma \cdot A - m]$. You will need the facts that the operator $\psi(x)$ obeys the Dirac equation in the external field $A(x)$, that $d\theta/dt = \delta(t)$ and that the anticommutation relations for $\psi(x), \overline{\psi}(y)$ at $x^0 = y^0$ are not changed by the presence of A.

Solution:

$$\frac{\langle out|\, T[\psi_\alpha(x)\bar{\psi}_\beta(y)]\, |in\rangle}{\langle out|in\rangle} \equiv \theta(x^0-y^0)\frac{\langle out|\, \psi_\alpha(x)\bar{\psi}_\beta(y)\, |in\rangle}{\langle out|in\rangle} - \theta(y^0-x^0)\frac{\langle out|\, \bar{\psi}_\beta(y)\psi_\alpha(x)\, |in\rangle}{\langle out|in\rangle}$$

Since $[i\gamma\cdot\partial + q\gamma\cdot A - m]\psi = 0$, applying this operator to both sides of equation gives on the right only contributions of the time derivative on the step functions:

$$\begin{aligned}
&[i\gamma\cdot\partial + q\gamma\cdot A - m]_{\gamma\alpha}\frac{\langle out|\, T[\psi_\alpha(x)\bar{\psi}_\beta(y)]\, |in\rangle}{\langle out|in\rangle} \\
&\quad = i\gamma^0_{\gamma\alpha}\delta(x^0-y^0)\frac{\langle out|\, (\psi_\alpha(x)\bar{\psi}_\beta(y) + \bar{\psi}_\beta(y)\psi_\alpha(x))\, |in\rangle}{\langle out|in\rangle} \\
&\quad = i\gamma^0_{\gamma\alpha}\delta(x^0-y^0)\frac{\langle out|\, \gamma^0_{\alpha\beta}\delta(\vec{x}-\vec{y})\, |in\rangle}{\langle out|in\rangle} \\
&\quad = i\delta_{\gamma\beta}\delta(x-y)
\end{aligned}$$

which is what was to be shown.

3. Pair production in an external EM field

   a) Write down the lowest order contribution to the amplitude for producing an electron-positron pair in the presence of an external field A_μ. Which Fourier components of $A_\mu(x)$ contribute to pair production to lowest order in qA? What will happen to the vacuum persistence amplitude $\langle out|in\rangle$ if pair production is possible?

   Solution:

$$\mathcal{M} = \langle 0|\, d_{\lambda_1}(\vec{q}_1)b_{\lambda_2}(\vec{q}_2)iQ\int d^4x A\cdot\bar{\psi}\gamma\psi\, |0\rangle = -ie\bar{u}_{\lambda_2}(\vec{q}_2)\gamma\cdot\tilde{A}(q_1+q_2)v_{\lambda_1}(\vec{q}_1).$$

   There can be pair production provided there are Fourier components $q = q_1 + q_2$ with $q^2 \leq -4m^2$, $q^0 > 0$. These are the conditions for $q = q_1 + q_2$ to be the four momentum of a particle antiparticle pair. Note that since the frequency $q^0 = \omega_1 + \omega_2 > 2m$ the conditions for adiabatic change are badly violated. If the probability of pair production is nonzero, the probability of vacuum persistence $|\langle out|in\rangle|^2 < 1$ (This could *not* happen for adiabatic change!)

   b) Calculate the differential probability of producing a pair with momenta $\mathbf{q}_1$, $\mathbf{q}_2$ and unobserved helicities, in terms of $\tilde{A}_\mu$, the Fourier transform of

$A_\mu(x)$. [You will need

$$\sum_\lambda u_\lambda(p)\overline{u}_\lambda(p) = m - \gamma\cdot p, \qquad \sum_\lambda v_\lambda(p)\overline{v}_\lambda(p) = -m - \gamma\cdot p$$

and the formulae for the trace of products of γ matrices. Remember that $\tilde{A}$ is complex and that $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$.] For the special case $\mathbf{q}_1 = \mathbf{q}$, $\mathbf{q}_2 = -\mathbf{q}$, $q_1^0 = q_2^0 = \omega(\mathbf{q})$, verify that your answer is positive and that it does not depend on $\tilde{A}^0$, and express it in terms of Fourier components of the electromagnetic field $F_{\mu\nu}$. [This verifies the gauge invariance of the calculation.]

:**Solution**

$$dP = \sum_{\lambda_1,\lambda_2} \frac{d^3q_1 d^3q_2}{(2\pi)^6 4\omega_1\omega_2}|\mathcal{M}|^2 = e^2\frac{d^3q_1 d^3q_2}{(2\pi)^6 4\omega_1\omega_2}\,\mathrm{Tr}\,\gamma\cdot\tilde{A}(-m-\gamma\cdot q_1)\gamma\cdot\tilde{A}^*(m-\gamma\cdot q_2)$$
$$= e^2\frac{d^3q_1 d^3q_2}{(2\pi)^6\omega_1\omega_2}\left(-\frac{(q_1+q_2)^2\tilde{A}\cdot\tilde{A}^*}{2} + q_1\cdot\tilde{A}q_2\cdot\tilde{A}^* + q_1\cdot\tilde{A}^*q_2\cdot\tilde{A}\right)$$

For $\vec{q}_1 = -\vec{q}_2 = \vec{q}$, $q_1^0 = q_2^0 = \omega$

$$dP = e^2\frac{d^3q_1 d^3q_2}{(2\pi)^6\omega^2}\left(2\omega^2\tilde{A}\cdot\tilde{A}^* - 2\vec{q}\cdot\vec{\tilde{A}}\vec{q}\cdot\vec{\tilde{A}}^* + 2\omega^2\tilde{A}^0\tilde{A}^{0*}\right)$$
$$= 2e^2\frac{d^3q_1 d^3q_2}{(2\pi)^6}\left(\vec{\tilde{A}}\cdot\vec{\tilde{A}}^* - \frac{\vec{q}}{\omega}\cdot\vec{\tilde{A}}\frac{\vec{q}}{\omega}\cdot\vec{\tilde{A}}^*\right)$$

this is > 0 because $|\vec{q}| < \omega$. The electric field $\vec{E} = -\nabla A^0 - \dot{\vec{A}} \to -i(\mathbf{q}_1 + \mathbf{q}_2)\tilde{A}^0 + 2i\omega\vec{\tilde{A}} = 2i\omega\vec{\tilde{A}}$ so $\vec{\tilde{A}} = \vec{\tilde{E}}/2i\omega$. Thus

$$dP = e^2\frac{d^3q_1 d^3q_2}{2(2\pi)^6\omega^2}\left(\vec{\tilde{E}}\cdot\vec{\tilde{E}}^* - \frac{\vec{q}}{\omega}\cdot\vec{\tilde{E}}\frac{\vec{q}}{\omega}\cdot\vec{\tilde{E}}^*\right)$$

c) The total probability for producing a pair is the integral of the result of part b) over $\mathbf{q}_1$, $\mathbf{q}_2$. This is a six dimensional integral but the external field depends only on the four components of $K = q_1 + q_2$. Integration over the remaining two variables can be carried out by the following procedure. Insert $1 = \int d^4K\delta(K - q_1 - q_2)$ into the integrand and replace the arguments of the fields by K. Then one needs the values of

$$F(K^2) \equiv \int \frac{d^3q_1}{2\omega_1}\frac{d^3q_2}{2\omega_2}\delta(K-q_1-q_2)$$
$$G^{\mu\nu}(K) \equiv \int \frac{d^3q_1}{2\omega_1}\frac{d^3q_2}{2\omega_2} q_1^\mu q_2^\nu \delta(K-q_1-q_2).$$

By considering G^μ_μ and $K_\mu K_\nu G^{\mu\nu}$ and Lorentz covariance, show that

$$G^{\mu\nu} = \left[\left(\frac{1}{6}-\frac{m^2}{3K^2}\right)K^\mu K^\nu + \left(\frac{K^2}{12}+\frac{m^2}{3}\right)\eta^{\mu\nu}\right]F(K^2).$$

Since $F(K^2)$ is a Lorentz invariant, it may be evaluated in any frame. By choosing the frame in which $\mathbf{K}=0$, show that

$$F(K^2) = \frac{\pi}{2}\sqrt{1+\frac{4m^2}{K^2}}\theta(K^0)\theta(-K^2-4m^2),$$

where the step functions simply reflect the fact that in the center of mass frame of the two particles, $q_1^0+q_2^0 \geq 2m$.

Solution: By Lorentz covariance $G^{\mu\nu} = AK^\mu K^\nu + B\eta^{\mu\nu}$. Now $K^2 = -2m^2+2q_1\cdot q_2$, so $G^\mu_\mu = AK^2+4B = (m^2+K^2/2)F(K^2)$. Also $K\cdot q_1 = q_1^2+q_1\cdot q_2 = K^2/2 = K\cdot q_2$. hence $K_\mu K_\nu G^{\mu\nu} = AK^4+BK^2 = K^4F(K^2)/4$. Solving gives $B=(m^2+K^2/4)F(K^2)/3$, $A=-(m^2-K^2/2)F(K^2)/(3K^2)$. Then

$$G^{\mu\nu} = \left[\left(\frac{1}{6}-\frac{m^2}{3K^2}\right)K^\mu K^\nu + \left(\frac{K^2}{12}+\frac{m^2}{3}\right)\eta^{\mu\nu}\right]F(K^2).$$

In the frame in which $\mathbf{K}=0$, $K^2=-K^{02}=-4\omega_1^2$

$$F(K^2) = \int \frac{d^3q_1}{4\omega_1^2}\delta(K^0-2\omega_1) = \pi\frac{q_1}{2\omega_1} = \frac{\pi}{2}\sqrt{1-\frac{m^2}{\omega_1^2}},$$
$$= \frac{\pi}{2}\sqrt{1+\frac{4m^2}{K^2}}\theta(K^0)\theta(-K^2-4m^2)$$

where the step functions simply reflect the fact that in the center of mass frame of the two particles, $q_1^0+q_2^0 \geq 2m$.

d) Using the results of part c), express the total probability for the production of a pair with unobserved momenta and helicities as an integral over K of an explicit function of K.

Solution Plugging in the above results gives

$$P = \frac{e^2}{6\pi} \int \frac{d^4K}{(2\pi)^4} \theta(K^0)\theta(-K^2-4m^2) \left(1 - \frac{2m^2}{K^2}\right) \sqrt{1 + \frac{4m^2}{K^2}} \left[K \cdot \tilde{A} K \cdot \tilde{A}^* - K^2 \tilde{A} \cdot \tilde{A}^*\right]$$

Finally note that $\tilde{F}_{\mu\nu} = iK_\mu \tilde{A}_\nu - iK_\nu \tilde{A}_\mu$ and compute $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu *} = 2(K^2\tilde{A} \cdot \tilde{A}^* - K \cdot \tilde{A} K \cdot \tilde{A}^*)$. So we can put our result in the gauge invariant form

$$P = \frac{\alpha}{3} \int \frac{d^4K}{(2\pi)^4} \theta(K^0)\theta(-K^2 - 4m^2) \left(1 - \frac{2m^2}{K^2}\right) \sqrt{1 + \frac{4m^2}{K^2}} \left[-\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu *}\right]$$

4. S, Problem 49.1

   a)

$$\text{h.c.} = e\sqrt{2}[E_L \psi^\dagger P_L \gamma^0 X + E_R \psi^\dagger P_R \gamma^0 X] = e\sqrt{2}[E_L \bar{\psi} P_R X + E_R \bar{\psi} P_L X]$$

   b) To lowest order, we need

$$\left\langle X(u)X(v)\psi(x)\bar{\psi}(y)\frac{i^2}{2}\left(\int d^4z \mathcal{L}(z)\right)^2\right\rangle$$

$$= -2e^2 \int d^4z d^4z' \left\langle X(u)X(v)\psi(x)\bar{\psi}(y)[E_L\bar{\psi}P_R X + E_R\bar{\psi}P_L X]_z [E_L^\dagger \bar{X} P_L \psi + E_R^\dagger \bar{X} P_R \psi]_{z'}\right\rangle$$

$$= -2e^2 \int d^4z d^4z' \Delta_F^L(z-z') \left\langle X(u)X(v)(-)S_F(x-z)(x)P_R X(z)\bar{X}(z')P_L S_F(z'-y)\right\rangle$$

$$-2e^2 \int d^4z d^4z' \Delta_F^R(z-z') \left\langle X(u)X(v)(-)S_F(x-z)(x)P_L X(z)\bar{X}(z')P_R S_F(z'-y)\right\rangle$$

where we have first contracted out the charged scalar and Fermi fields. Here the S_F's use the electron mass. To contract out the X's we need $\left\langle X(u)\bar{X}(z)\right\rangle = S_F^X(u-z)$ and $\left\langle X(u)X^T(z)\right\rangle = S_F^X(u-z)C$, with $C = i\gamma^2\gamma^0$. Notice that the second line is just the first line with $L \leftrightarrow R$. Thus we only write out the contractions for the first line:

$$\text{1st line} = 2e^2 \int d^4z d^4z' \Delta_F^L(z-z')[S_F(x-z)P_R S_F^X(z-u)C\ S_F^X(v-z')P_L S_F(z'-y)$$

$$-S_F(x-z)P_R S_F^X(z-v)C\ S_F^X(u-z')P_L S_F(z'-y)]$$

Going to momentum space and applying the reduction rules then leads to

$$\mathcal{M} = -2ie^2\left(\frac{\bar{v}(p_+)P_R v_X(q_1)\bar{u}_X(q_2)P_L u(p_-)}{M_L^2+(q_1-p_+)^2} - \frac{\bar{v}(p_+)P_R v_X(q_2)\bar{u}_X(q_1)P_L u(p_-)}{M_L^2+(q_2-p_+)^2}\right)$$

$$-2ie^2\left(\frac{\bar{v}(p_+)P_L v_X(q_1)\bar{u}_X(q_2)P_R u(p_-)}{M_R^2+(q_1-p_+)^2} - \frac{\bar{v}(p_+)P_L v_X(q_2)\bar{u}_X(q_1)P_R u(p_-)}{M_R^2+(q_2-p_+)^2}\right)$$

c) When we square and sum over all spins, the terms coming from multiplying the first line with the second line contribute zero because we are treating $m_e = 0$ and the chiral projectors will then come in the form $P_L\gamma\cdot p_\pm P_L = P_L P_R \gamma\cdot p_\pm = 0$. Thus, assuming $t, u \ll M_L^2 = M_R^2$,

$$\left\langle|\mathcal{M}|^2\right\rangle = \frac{1}{4}\frac{4e^4}{M_L^4}[4p_-\cdot q_2 p_+\cdot q_1 + 4p_-\cdot q_1 p_+\cdot q_2$$

$$-4\left\langle\bar{u}_X(q_1)P_L(-\gamma\cdot p_-)u_X(q_2)\bar{v}_X(q_1)P_L(-\gamma\cdot p_+)v_X(q_2)] + (q_1\leftrightarrow q_2)\right\rangle + (L\leftrightarrow R)]$$

It is simple to show that $\bar{v}_X(q_1)P_L(-\gamma\cdot p_+)v_X(q_2) = \bar{u}_X(q_2)P_R(-\gamma\cdot p_+)u_X(q_2)$. Thus the spin sums on the second line can also be replaced by traces

$$\left\langle|\mathcal{M}|^2\right\rangle = \frac{1}{4}\frac{4e^4}{M_L^4}[4p_-\cdot q_2 p_+\cdot q_1 + 4p_-\cdot q_1 p_+\cdot q_2$$

$$-2\,\mathrm{Tr}\,(M_X-\gamma\cdot q_1)P_L(-\gamma\cdot p_-)(M_x-\gamma\cdot q_2)P_R(-\gamma\cdot p_+) + (L\leftrightarrow R)]$$

$$= \frac{1}{4}\frac{4e^4}{M_L^4}[8p_-\cdot q_2 p_+\cdot q_1 + 8p_-\cdot q_1 p_+\cdot q_2 - 2M_X^2\,\mathrm{Tr}\,(-\gamma\cdot p_-)(-\gamma\cdot p_+)]$$

$$= \frac{2e^4}{M_L^4}[4p_-\cdot q_2 p_+\cdot q_1 + 4p_-\cdot q_1 p_+\cdot q_2 + 4M_X^2 p_-\cdot p_+]$$

$$= \frac{2e^4}{M_L^4}[(M_X^2-t)^2 + (M_X^2-u)^2 - 2M_X^2 s] = \frac{e^4}{M_L^4}s(s-4M_X^2)(1+\cos^2\theta)$$

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 s}\sqrt{1-\frac{4M_X^2}{s}} = \frac{e^4 s}{64\pi^2 M_L^4}\left(1-\frac{4M_X^2}{s}\right)^{3/2}(1+\cos^2\theta)$$