Quantum Field Theory I

Problem Set 8

Due: 26 November 2007

1. Calculate the differential cross-section for electron scattering in the field of a magnetic dipole when the initial and final momenta are both perpendicular to the dipole axis, to lowest order in perturbation theory (Born approximation). [The vector potential of a dipole $\vec{M}$ is $\nabla \times \left(\vec{M}/r\right)$ so you can find its Fourier transform from that of $1/r$.]

Solution: Writing the amplitude, $\mathcal{M} = -ie\tilde{\vec{A}}(\vec{q}) \cdot \bar{u}'\vec{\gamma}u$, where $q = p' - p$. So we need to Fourier transform the vector potential

$$\tilde{\vec{A}} = \int d^3x e^{-i\vec{q}\cdot\vec{x}} \frac{\nabla \times \vec{M}}{r} = i\vec{q} \times \vec{M} \int d^3x \frac{e^{-i\vec{q}\cdot\vec{x}}}{r} = 4\pi i \frac{\vec{q} \times \vec{M}}{\vec{q}^2}$$

Then

$$\mathcal{M} = 4\pi e \frac{\vec{q} \times \vec{M}}{\vec{q}^2} \cdot \bar{u}'\vec{\gamma}u = \frac{16\pi e\lambda\delta_{\lambda'\lambda}}{\vec{q}^2} \chi_\lambda^\dagger(\vec{p}')\vec{\sigma} \cdot (\vec{q} \times \vec{M})\chi_\lambda(\vec{p})$$

Because of helicity conservation we may assume $\lambda' = \lambda = \pm 1/2$ and find using the projector trick

$$\begin{aligned}
|\mathcal{M}|^2 &= \frac{64\pi^2 e^2 \vec{p}^2}{\vec{q}^4} \mathrm{Tr}\left[\vec{\sigma} \cdot (\vec{q} \times \vec{M}) \frac{1 + 2\lambda\hat{p}' \cdot \sigma}{2} \vec{\sigma} \cdot (\vec{q} \times \vec{M}) \frac{1 + 2\lambda\hat{p} \cdot \sigma}{2}\right] \\
&= \frac{64\pi^2 e^2 \vec{p}^2}{\vec{q}^4} \mathrm{Tr}\left[(\vec{q} \times \vec{M})^2 \frac{1 - 2\lambda\hat{p}' \cdot \sigma}{2} \frac{1 + 2\lambda\hat{p} \cdot \sigma}{2} + 2\lambda\hat{p}' \cdot (\vec{q} \times \vec{M})\vec{\sigma} \cdot (\vec{q} \times \vec{M}) \frac{1 + 2\lambda\hat{p} \cdot \sigma}{2}\right] \\
&= \frac{64\pi^2 e^2 \vec{p}^2}{\vec{q}^4} \left[(\vec{q} \times \vec{M})^2 \frac{1 - \hat{p}' \cdot \hat{p}}{2} + \vec{p}^2[(\hat{p} \times \hat{p}') \cdot \vec{M}]^2\right]
\end{aligned}$$

For $\vec{p}, \vec{p}'$ perpendicular to $\vec{M}$, $(\vec{q} \times \vec{M})^2 = \vec{q}^2 \vec{M}^2$ and $[(\hat{p} \times \hat{p}') \cdot \vec{M}]^2 = \vec{M}^2 \sin^2\phi$, with ϕ the scattering angle. Also $\vec{q}^2 = 2\vec{p}^2(1 - \cos\phi)$, so

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{16\pi^2} = \frac{e^2\vec{M}^2}{(1 - \cos\phi)^2}\left[(1 - \cos\phi)^2 + \sin^2\phi\right] = \frac{e^2\vec{M}^2}{\sin^2(\phi/2)}$$

2. S, Problem 48.2 **Solution**: The amplitude for this process is

$$\bar{v}(q) ig \left[\frac{m - (p - k_2) \cdot \gamma}{m^2 + (p - k_2)^2} + \frac{m - (p - k_1) \cdot \gamma}{m^2 + (p - k_1)^2}\right] igu(p) = \bar{v}(q) ig \left[\frac{2m + k_2 \cdot \gamma}{m^2 - u} + \frac{2m + k_1 \cdot \gamma}{m^2 - t}\right] igu(p)$$

Then the spin averaged squared amplitude is

$$\left\langle|\mathcal{M}|^2\right\rangle = \frac{g^4}{4}\,\mathrm{Tr}(-m - q\cdot\gamma)B(m - p\cdot\gamma)B$$

where

$$B = \frac{2m + k_2\cdot\gamma}{m^2 - u} + \frac{2m + k_1\cdot\gamma}{m^2 - t}$$

If we identify $k_2 = k'$, $k_1 = -k$, $q = -p'$, $t = s$, we see that the right side becomes identical to $(-1/2)\times$ (48.18). The extra 1/2 is because both fermions have their spins averaged. Thus we can write

$$\left\langle|\mathcal{M}|^2\right\rangle = g^4\left[\frac{\langle\Phi_{tt}\rangle}{(m^2-t)^2} + \frac{\langle\Phi_{uu}\rangle}{(m^2-u)^2} + \frac{\langle\Phi_{tu}\rangle + \langle\Phi_{ut}\rangle}{(m^2-t)(m^2-u)}\right]$$

where

$$\begin{aligned}
\langle\Phi_{tt}\rangle &= -\frac{1}{2}(-tu + m^2(9t+u) + 7m^4 - 8m^2M^2 + M^4)\\
\langle\Phi_{uu}\rangle &= -\frac{1}{2}(-tu + m^2(9u+t) + 7m^4 - 8m^2M^2 + M^4)\\
\langle\Phi_{tu}\rangle &= -\frac{1}{2}(+tu + 3m^2(u+t) + 9m^4 - 8m^2M^2 - M^4)\\
\langle\Phi_{ut}\rangle &= -\frac{1}{2}(+tu + 3m^2(u+t) + 9m^4 - 8m^2M^2 - M^4)
\end{aligned}$$

3. S, Problem 48.5 **Solution**:

a) The amplitude for the decay is

$$\mathcal{M} = ic_1 G_F f_\pi \bar{u}(p)\gamma^\mu(1-\gamma_5)v(q)(ik_\mu)$$

where p, q, k are the muon, neutrino, and pion momenta respectively. The spin summed squared amplitude is then

$$\begin{aligned}
\left\langle|\mathcal{M}|^2\right\rangle &= c_1^2 G_F^2 f_\pi^2\,\mathrm{Tr}\,k\cdot\gamma(1-\gamma_5)(-q\cdot\gamma)k\cdot\gamma(1-\gamma_5)(m-\gamma\cdot p)\\
&= 2c_1^2 G_F^2 f_\pi^2\,\mathrm{Tr}\,k\cdot\gamma(1-\gamma_5)(-q\cdot\gamma)k\cdot\gamma(m-\gamma\cdot p)\\
&= 8c_1^2 G_F^2 f_\pi^2(2k\cdot q k\cdot p - k^2 q\cdot p) = 4c_1^2 G_F^2 f_\pi^2(m_\pi^2 - m_\mu^2)m_\mu^2
\end{aligned}$$

In the pion rest frame,

$$\begin{aligned}
\Gamma &= \int \frac{d\omega q dq}{32\pi^2 m_\pi E_\mu}\delta(m_\pi - q - \sqrt{m_\mu^2 + q^2})\left\langle|\mathcal{M}|^2\right\rangle = \frac{m_\pi^2 - m_\mu^2}{16\pi m_\pi^3}\left\langle|\mathcal{M}|^2\right\rangle\\
&= \frac{c_1^2 G_F^2 f_\pi^2 m_\mu^2(m_\pi^2 - m_\mu^2)^2}{4\pi m_\pi^3}
\end{aligned}$$

b) $\Gamma = \hbar/\tau_\pi = (6.58212/2.603) \times 10^{-14}\text{MeV}$. Then

$$f_\pi = \frac{\sqrt{4\pi m_\pi^3 (6.58212/2.603)} \times 10^{-7}}{c_1 G_F m_\mu (m_\pi^2 - m_\mu^2)} \approx 93.14\text{MeV}$$

4. We shall soon find, when we complete the quantization of the electromagnetic field, that the photon propagator can be taken to be

$$\langle A_\mu(x) A_\nu(y) \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-i\eta_{\mu\nu} e^{i(x-y)\cdot p}}{p^2 - i\epsilon}.$$

Anticipatiing this result, calculate, to lowest order, the differential cross-section for (a) e^+e^- scattering (Bhabha scattering) and (b) e^-e^- scattering (Moller scattering) for the case that both particles in the initial state are unpolarized and the final spins are unobserved. Notice that the expressions for the squares of the amplitudes summed over spins for these two processes are simply related to each other by changing the signs and interchanging some of the 4-momenta. Thus the traces need to be calculated for only one of the two processes. Give the differential cross section for both processes in the Center of Mass frame. Discuss the limiting cases of low and high energy for both processes.

Solution: The amplitude for the two diagrams for e^+e^- scattering is

$$\mathcal{M}_{\text{Bhabba}} = -ie^2 \left[\frac{\bar{v}\gamma_\mu v' \bar{u}' \gamma^\mu u}{t} - \frac{\bar{u}'\gamma_\mu v' \bar{v} \gamma^\mu u}{s} \right]$$

where the primed spinors refer to final state particles and unprimed to initial state particles. Let p, p' be electron momenta and q, q' positron momenta, so $t = -(q' - q)^2$ and $s = -(q + p)^2$. Summing over all particle spins converts $|\mathcal{M}|^2$ to traces:

$$\begin{aligned}
\Sigma|\mathcal{M}|^2 = e^4 \Bigg[&\frac{1}{t^2} \text{Tr}\{\gamma_\mu(m + q' \cdot \gamma)\gamma_\lambda(m + q \cdot \gamma)\} \, \text{Tr}\{\gamma^\mu(m - p \cdot \gamma)\gamma^\lambda(m - p' \cdot \gamma)\} \\
&+ \frac{1}{s^2} \text{Tr}\{\gamma_\mu(m + q' \cdot \gamma)\gamma_\lambda(m - p' \cdot \gamma)\} \, \text{Tr}\{\gamma^\mu(m - p \cdot \gamma)\gamma^\lambda m + q \cdot \gamma)\} \\
&- \frac{1}{st} \text{Tr}\{\gamma_\mu(m + q' \cdot \gamma)\gamma_\lambda(m + q \cdot \gamma)\gamma^\mu(m - p \cdot \gamma)\gamma^\lambda(m - p' \cdot \gamma)\} \\
&- \frac{1}{st} \text{Tr}\{\gamma_\mu(m + q' \cdot \gamma)\gamma_\lambda(m - p' \cdot \gamma)\gamma^\mu(m - p \cdot \gamma)\gamma^\lambda(m + q \cdot \gamma)\} \Bigg]
\end{aligned}$$

We list some intermediate trace results:

$$
\begin{aligned}
&\mathrm{Tr}\{\gamma_\mu q'\cdot\gamma\gamma_\lambda q\cdot\gamma\}\,\mathrm{Tr}\{\gamma^\mu p\cdot\gamma\gamma^\lambda p'\cdot\gamma\} = 16(q'_\mu q_\lambda + q'_\lambda q_\mu - \eta_{\mu\lambda} q\cdot q')(p'_\mu p_\lambda + p'_\lambda p_\mu - \eta_{\mu\lambda} p\cdot p')\\
&= 32(q'\cdot p' q\cdot p + q'\cdot p q\cdot p')\\
&= 8((s-2m^2)^2 + (s+t-2m^2)^2) = 16s^2 + 16st + 8t^2 - 32m^2(2s+t) + 64m^4\\
&\mathrm{Tr}\{\gamma_\mu q'\cdot\gamma\gamma_\lambda p'\cdot\gamma\}\,\mathrm{Tr}\{\gamma^\mu p\cdot\gamma\gamma^\lambda q\cdot\gamma\} = 16t^2 + 16st + 8s^2 - 32m^2(2t+s) + 64m^4\\
&\mathrm{Tr}\{\gamma_\mu q'\cdot\gamma\gamma_\lambda q\cdot\gamma\gamma^\mu p\cdot\gamma\gamma^\lambda p'\cdot\gamma\} = \mathrm{Tr}\{\gamma_\mu q'\cdot\gamma\gamma_\lambda p'\cdot\gamma\gamma^\mu p\cdot\gamma\gamma^\lambda q\cdot\gamma\}\\
&= 2\,\mathrm{Tr}\{q\cdot\gamma\gamma_\lambda q'\cdot\gamma p\cdot\gamma\gamma^\lambda p'\cdot\gamma\}\\
&= 8q'\cdot p\,\mathrm{Tr}\{q\cdot\gamma p'\cdot\gamma\} = -32 q'\cdot p q\cdot p' = -8(s+t-2m^2)^2
\end{aligned}
$$

Putting everything together and simplifying,

$$\Sigma|\mathcal{M}|^2 = 16e^4\left[\frac{s^2}{t^2} + \frac{t^2}{s^2} + 2\frac{s}{t} + 2\frac{t}{s} + 3 - 4m^2\left(\frac{s}{t^2} + \frac{t}{s^2}\right) + 4\left(\frac{m^4}{t^2} + \frac{m^4}{s^2} - \frac{m^4}{st}\right)\right]$$

Averaging over initial spins means to divide this sum by 4. Then plugging into the cross section formula we get

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{e^4}{16\pi^2 s}\left[\frac{s^2}{t^2} + \frac{t^2}{s^2} + 2\frac{s}{t} + 2\frac{t}{s} + 3 - 4m^2\left(\frac{s}{t^2} + \frac{t}{s^2}\right) + 4\left(\frac{m^4}{t^2} + \frac{m^4}{s^2} - \frac{m^4}{st}\right)\right]\\
&= \frac{\alpha^2}{s}\left[\frac{s^2}{t^2} + \frac{t^2}{s^2} + 2\frac{s}{t} + 2\frac{t}{s} + 3 - 4m^2\left(\frac{s}{t^2} + \frac{t}{s^2}\right) + 4\left(\frac{m^4}{t^2} + \frac{m^4}{s^2} - \frac{m^4}{st}\right)\right]
\end{aligned}
$$

$t = -(p-p')^2 = -2\vec{p}^2(1-\cos\theta) = -(s-4m^2)\sin^2(\theta/2)$. Thus at low energy $-t \ll s$ and

$$\frac{d\sigma}{d\Omega} \sim \frac{\alpha^2 m^2}{16|\vec{p}|^4\sin^4(\theta/2)}$$

which is just the non-relativistic Rutherford scattering result. On the other hand at high energy we can neglect the electron mass and obtain

$$\frac{d\sigma}{d\Omega} \sim \frac{\alpha^2}{s}\left[\frac{1}{\sin^4(\theta/2)} + \sin^4(\theta/2) - \frac{2}{\sin^2(\theta/2)} - 2\sin^2(\theta/2) + 3\right]$$

The $\theta \to 0$ divergence comes from the $1/t$ in the photon exchange diagram.

For e^-e^- scattering, we see that the spin summed square of the amplitude can be obtrained from that for e^+e^- by substituting $s \to u$:

$$\Sigma|\mathcal{M}|^2 = 16e^4 \left[\frac{u^2}{t^2} + \frac{t^2}{u^2} + 2\frac{u}{t} + 2\frac{t}{u} + 3 - 4m^2\left(\frac{u}{t^2} + \frac{t}{u^2}\right) + 4\left(\frac{m^4}{t^2} + \frac{m^4}{u^2} - \frac{m^4}{ut}\right)\right]$$

so the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s}\left[\frac{u^2}{t^2} + \frac{t^2}{u^2} + 2\frac{u}{t} + 2\frac{t}{u} + 3 - 4m^2\left(\frac{u}{t^2} + \frac{t}{u^2}\right) + 4\left(\frac{m^4}{t^2} + \frac{m^4}{u^2} - \frac{m^4}{ut}\right)\right]$$

Now $u = -(s-4m^2)\cos^2(\theta/2)$, giving $u/t = \cot^2(\theta/2)$. At low energies $-u, -t \ll s \approx 4m^2$ so at fixed angle the last term dominates

$$\frac{d\sigma}{d\Omega} \sim \frac{\alpha^2 m^2}{16|\vec{p}|^2}\frac{1 - 3\sin^2(\theta/2)\cos^2(\theta/2)}{\sin^4(\theta/2)\cos^4(\theta/2)} = \frac{\alpha^2 m^2}{4|\vec{p}|^2}\frac{1 + 3\cos^2\theta}{\sin^4\theta}$$

At high energy we can neglect the electron mass obtaining

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &\sim \frac{\alpha^2}{s}\left[\frac{u^2}{t^2} + \frac{t^2}{u^2} + 2\frac{u}{t} + 2\frac{t}{u} + 3\right] \\
&\sim \frac{\alpha^2}{s}\left[\cot^4(\theta/2) + \tan^4(\theta/2) + 2\cot^2(\theta/2) + 2\tan^2(\theta/2) + 3\right] \\
&\sim \frac{\alpha^2}{s}\left[\frac{1}{\sin^4(\theta/2)} + \frac{1}{\cos^4(\theta/2)} + 1\right]
\end{aligned}$$