Standard Model/Quantun Field Theory

Problem Set 2

Due: Monday, 23 September 2019

Suggested reading: QFT Notes Chs 18-19; Textbook: Secs. 62-65.

5. As discussed in class, gauge invariance in QED implies the Ward identity

$$q_\mu \Gamma^\mu(p', p) = m + \gamma \cdot p' + \Sigma(p') - (m + \gamma \cdot p + \Sigma(p)) \tag{1}$$

where $q = p' - p$. Sandwiching both sides with on **physical** mass shell spinors reduces this statement to $q_\mu \bar{u}' \Gamma^\mu u = 0$.

a) Show that the quantities

$$\bar{u}'\gamma^\mu u, \quad \bar{u}'(p'+p)^\mu u, \quad \bar{u}'\sigma^{\mu\nu} q_\nu u$$

satisfy this conservation law (recall that $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$).

b) Prove the identity

$$2i\bar{u}'\sigma^{\mu\nu} q_\nu u = 2\bar{u}'(p'+p)^\mu u - 4m\bar{u}'\gamma^\mu u.$$

And argue that the most general Lorentz and parity covariant, and gauge invariant form for Γ^μ, when sandwiched between on physical mass shell spinors, can be taken to be

$$\Gamma^\mu = \gamma^\mu F_1(q^2) - \frac{i}{2m}\sigma^{\mu\nu} q_\nu F_2(q^2).$$

6. In class we calculated the proper vertex function in terms of two Feynman parameters:

$$\bar{u}'\Gamma^\mu(p', p)u \equiv \bar{u}'\left[\gamma^\mu F_1(q^2) + \frac{[\gamma^\mu, q\cdot\gamma]}{4m} F_2(q^2)\right] u \tag{2}$$

$$F_1(q^2) = 1 + \frac{Q_0^2}{8\pi^2}\int dxdy\Bigg(\ln \frac{\Lambda^2}{\lambda^2(1-x-y) + m^2(x+y)^2 + xyq^2} - \frac{3}{2} + \frac{m^2((x+y)^2 - 2(1-x-y) - q^2(1-x)(1-y))}{\lambda^2(1-x-y) + m^2(x+y)^2 + xyq^2}\Bigg) \tag{3}$$

$$F_2(q^2) = \frac{Q_0^2}{4\pi^2}\int dxdy \frac{m^2(x+y)(1-x-y)}{\lambda^2(1-x-y) + m^2(x+y)^2 + xyq^2} \tag{4}$$

where λ is a small photon mass introduced as an infrared cutoff. Also the range of integration is $x + y \leq 1$ In this problem we study the limit $\lambda \to 0$.

a) First make a convenient change of variables $x = u(1-v)$, $y = uv$. show that the range of integration for the new variables is $0 < u, v < 1$ and that

$$F_1(q^2) = 1 + \frac{Q_0^2}{8\pi^2}\int dvudu\Bigg(\ln \frac{\Lambda^2}{\lambda^2(1-u) + u^2(m^2 + v(1-v)q^2)} - \frac{3}{2} + \frac{m^2(u^2 - 2(1-u)) - q^2(1-u+u^2v(1-v))}{\lambda^2(1-u) + m^2u^2 + q^2u^2v(1-v)}\Bigg) \tag{5}$$

$$F_2(q^2) = \frac{Q_0^2}{4\pi^2}\int dvudu \frac{m^2u(1-u)}{\lambda^2(1-u) + u^2(m^2 + v(1-v)q^2)} \tag{6}$$

b) Show that one may safely set $\lambda = 0$ in the integrand of F_2 and in all the terms in the integrand of F_1, except for the terms in the numerator of the last term which have no factors of u. For all of the terms allowing $\lambda = 0$, the u integral is elementary so do each one!

c) This leaves all the λ dependence in the integral

$$\int dvudu \frac{-2m^2 - q^2}{\lambda^2(1-u) + u^2(m^2 + q^2 v(1-v))}; \tag{7}$$

which would be log divergence with $\lambda = 0$, so we expect there to be a $\ln \lambda$ dependence. One way to extract it is to break the u integration range into two regions: Pick an ϵ satisfying $\lambda/m \ll \epsilon \ll 1$. For $0 < u < \epsilon$ show that it is valid to replace the denominator by $\lambda^2 + u^2(m^2 + q^2 v(1-v))$ and then do the u integral. For $\epsilon < u < 1$ set $\lambda = 0$ (why valid?) and do the integral. Show that the dependence on ϵ cancels between the two terms.

d) Put everything together expressing F_1 and F_2 as single integrals over v, showing the explicit $\ln \Lambda$ and $\ln \lambda$ terms. The final result is accurate up to terms that vanish as $\lambda \to 0$.

7. **Bremsstrahlung.** We want to calculate, in tree approximation. the cross-section for scattering of an electron in a static external field with emission of a photon, in particular a soft photon. This problem steps you through calculations sketched in class filling in some missing details.

a) In lowest order in the external field there are two graphs. Write down the amplitude for the scattering of an electron of 4-momentum p_1 to a final state p_2 with emission of a photon of momentum k and polarization ϵ in a static field $A^\mu_{\rm ext}(\mathbf{x})$.

b) Take as variables the initial momentum $\mathbf{p_1}$, the magnitude and direction of photon momentum k, $\hat{\mathbf{k}}$, and the direction of the final electron momentum $\hat{\mathbf{p}}_2$. Write down an expression for the differential cross-section in terms of the amplitude defined in (a), summed and averaged over electron spins.

c) As $|\mathbf{k}| \to 0$, the cross-section behaves like dk/k (so that the total cross-section diverges)! Identify the terms in the amplitude that give rise to this behavior and calculate the coefficient of dk/k in $d\sigma$ with $k = 0$.

d) Verify the gauge invariance of this leading term in the amplitude as $k \to 0$, *i.e.* that if you substitute k_μ for ϵ_μ the amplitude vanishes.

e) Suppose we try to calculate to all orders in $A_{\rm ext}$. In each order, identify two graphs which will lead to a dk/k term in $d\sigma$ (do not try to show there are only two, though it is true). Show that the sum of all these graphs as $k \to 0$ is related to the amplitude for elastic scattering in the field $A_{\rm ext}$ and find the expression for $d\sigma$ in terms of $(d\sigma/d\Omega)_{\rm elastic}$.

f) Assume that the sum of all the other graphs not singled out in part (e) has a well-defined (*i.e.* independent of photon angles $\hat{\mathbf{k}}$) limit as $k \to 0$. Then this sum will depend on $\mathbf{p}_1$ and $\hat{\mathbf{p}}_2$ only. The amplitude coming from the graphs in (e) can be expanded in powers of k, the coefficients of each power depending on $\mathbf{p}_1$, $\hat{\mathbf{p}}_2$, $\hat{\mathbf{k}}$. Show that gauge-invariance (as stated in (d)) relates the term independent of k coming from the graphs singled out in (e) to the $k = 0$ limit of the rest of the amplitude. Show that this means we can find the term in $d\sigma/dk$ independent of k from the graphs in (e) only.

The result in part (e) seems, at first glance, to involve derivatives of the elastic amplitude w.r.t. the momenta in off-shell directions, which would require information not present in the elastic (on-shell) scattering amplitude. In fact, one can show that the derivatives in these off-shell directions cancel in the total amplitude. Thus not only the $O(1/k)$ **but also the** $O(k^0)$ **terms** in the bremstrahlung cross section are completely determined by the exact elastic electron scattering amplitude. This remarkable property of low energy photon emission was first discovered by F. E. Low, Physical Review **110** (1958) 974.