Standard Model/Quantun Field Theory

Problem Set 3

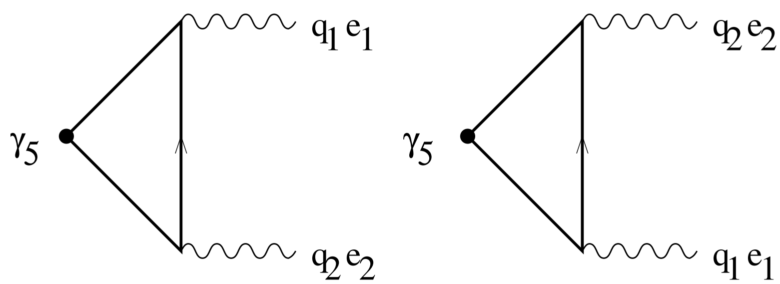
Due: 7 October 2019

Suggested reading: QFT Notes Chs 19-20; Textbook: Secs. 65; 67-68

8. As we shall see later, applying the methods for constructing a gauge invariant EM current to the axial vector current $j_5^\mu = \bar{\psi}\gamma_5\gamma^\mu\psi$ leads to the anomalous conservation law

$$\partial_\mu j_5^\mu(x) = -2m\bar{\psi}\gamma_5\psi + \frac{Q_0^2}{16\pi^2}\epsilon^{\mu\rho\nu\sigma}F_{\mu\rho}F_{\nu\sigma} \tag{1}$$

which is not zero even if $m = 0$. In this problem we shall confirm this result by explicitly calculating the matrix element $\langle 0|\bar{\psi}(x)\gamma_5\psi(x)|q_1e_1;\, q_2e_2\rangle$ of the pseudoscalar operator $\bar{\psi}(x)\gamma_5\psi(x)$ between the vacuum and a 2-photon state to lowest order in QED perturbation theory. The $e_{1,2}$ are the photon polarization vectors and $q_{1,2}$ are the photon momenta. Let $q = q_1 + q_2$. The x-dependence is of course $e^{iq\cdot x}$. You will have to evaluate the two Feynman graphs,



a) The trace calculation is not too difficult because

$$\mathrm{Tr}\gamma_5\gamma\cdot A\gamma\cdot B\gamma\cdot C\gamma\cdot D = -4i\epsilon^{\mu\nu\rho\sigma}A_\mu B_\nu C_\rho D_\sigma$$

Denote this quantity by [ABCD] and remember that it is antisymmetric. All other traces of γ_5 times less than 6γs are zero. Show that after taking the traces the integrals are convergent. Use the Feynman trick

$$\frac{1}{abc} = 2\int_{x,y,z>0} \frac{dxdydz\delta(1-x-y-z)}{(ax+by+cz)^3}$$

to express the matrix elements as $[q_1q_2e_1e_2]F(q^2)$ and find $F(0)$.

b) The same matrix element of the axial vector current, $J_5^\mu = \bar{\psi}(x)\gamma_5\gamma^\mu\psi(x)$, has a "naive divergence of $\partial_\mu J_g^\mu = (-2mi)$ times the above matrix element. The Pauli-Villars regularization of the axial current matrix element amounts to subtracting the value of the triangle graph with a large mass M from the value with the physical mass m. The r.h.s. of the divergence equation would therefore have a similar subtraction of $-2Mi$ times the result of part a) with $m \to M$. Show that this subtraction term as $M \to \infty$ gives precisely the anomalous term by comparing it to the same matrix element of the anomaly.

9. The construction of gauge covariant currents in the nonabelian case requires a generalization of the path dependent phase $\exp[iq\int d\xi^\mu A_\mu]$ to a matrix $P[\exp[ig\int d\xi^\mu A_\mu]]$ where the P denotes path ordering analogous to time ordering.

(a) Consider a fixed curve described by $\xi^\mu(t)$, with $0 \leq t \leq 1$. Call the path ordered phase, for the segment of the curve from 0 to T, $W(T)$. Show that W satisfies the differential equation

$$\frac{dW}{dT} = ig\frac{d\xi^\mu}{dt}(T)A_\mu(\xi(T))W.$$

(b) For a nonabelian gauge transformation $\Omega(\xi)$, show that $W_\Omega = \Omega(\xi(T))\; W\Omega^\dagger(\xi(0))$ satisfies the equation of part (a) with gauge field

$$\Omega A_\mu \Omega^\dagger - (i/g)(\partial_\mu \Omega)\Omega^\dagger,$$

which is just the nonabelian gauge transformation of A.

(c) Now prove the "nonabelian" Stokes theorem for an infinitesimal closed curve $\xi(1) = \xi(0)$:

$$Pe^{ig\oint d\xi^\mu A_\mu} \approx I + ig\int d\sigma^{\mu\nu}F_{\mu\nu}, \qquad \text{for } C \text{ infinitesimal},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$.

To deepen your understanding of the anomaly, I strongly recommend that you use the tools developed in this exercise to establish the results quoted in the footnotes of the notes for the axial anomaly in the nonabelian case.