Standard Model/Quantun Field Theory

Problem Set 6

Due: 15 November 2019

Suggested reading: QFT Notes Ch 23-24; Textbook: Secs. 83-84.

18. Suppose for a particular definition of the renormalized coupling constant, the power series for $\beta(g)$ is known

$$\beta(g) \quad = \quad b_1 g^3 + b_2 g^5 + b_3 g^7 + \cdots \tag{1}$$

In class we showed that with a redefinition of the renormalized coupling given as a power series

$$g'(g) = g + a_1 g^3 + a_2 g^5 + a_3 g^7 + \cdots, \tag{2}$$

the new beta function $\beta'(g')$ has a power series expansion where the first two terms $b_1 g'^3 + b_2 g'^5$ have coefficients identical to the old beta function.

a) Show, however, that the coefficient of g'^7 is changed, and further that there is a choice of the a_i which sets it to zero.

b) Argue that the a_i can be chosen so that the coefficients of all terms beyond the first two in β' are zero. In other words, there is a definition of the renormalized coupling such that the two loop beta function is the complete beta function! This was first noted by 't Hooft.

c) Assuming this has been done, solve the renormalization group equation exactly to get $t(g)$ as a function of g. Invert it (iteratively as $t \to \infty$ to get $g(t)$ as a function of t keeping terms of order $1/t^2$ and $(\ln t)/t^2$. as $t \to \infty$.

19. The result we quoted for the QCD beta function (Notes Eq(24.10)

$$\beta(g) \quad = \quad -\frac{Ng^3}{16\pi^2}\left(\frac{11}{3} - \frac{2N_f}{3N}\right) \tag{3}$$

includes the effects of the quark loop on the gluon propagator (N is the number of colors (3 for QCD), and N_f is the number of quark flavors (6) to date). We did not do this part of the calculation in class. By adapting the QED vacuum polarization calculation to QCD, complete the calculation of the beta function and confirm the correctness of the quoted result.

20. Confirm our expressions for $\gamma_G, \gamma_q, \beta$ in Notes section 24.2 for the gauge $\xi = 0$, by plugging our Landau gauge one loop evaluations of $G^{0,2}$ (gluon propagator), $G^{2,0}$ (quark propagator), and $G^{1,2}$ quark gluon vertex, Notes section 22.4, into the Callan-Symanzik equation.

21. **Muon Decay** The detailed study of the process $\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$ was crucial to establishing the current-current structure of the weak interactions, but in modern terms it represents the cleanest measurement of G_F.

a) Starting with the Feynman rules of the Standard Model Lagrangian, show that the tree amplitude for this process is, to a very good approximation (note that all momenta are of $O(.1\text{GeV}) \ll M_W$!)

$$\mathcal{M} = i\frac{G_F}{\sqrt{2}}\bar{u}_{\nu_\mu}\gamma^\lambda(1-\gamma_5)u_\mu\bar{u}_e\gamma_\lambda(1-\gamma_5)v_{\bar{\nu}_e} \tag{4}$$

b) Since the neutrinos are unobservable, it is convenient to rearrange the spinors in this expression so that the neutrino variables are in the same factor. Prove the Fierz rearrangement identity:

$$[\gamma^\mu(1-\gamma_5)]_{\alpha\beta}[\gamma_\mu(1-\gamma_5)]_{\gamma\delta} = -[\gamma^\mu(1-\gamma_5)]_{\alpha\delta}[\gamma_\mu(1-\gamma_5)]_{\gamma\beta}. \tag{5}$$

[Hint: use the fact that any 4×4 matrix can be written as a linear combination of the 16 matrices $I, \gamma_5, \gamma^\mu, \sigma^{\mu\nu}$.] Then the neutrino phase space integral involves

$$\int \frac{d^3q_1 d^3q_2}{4|q_1||q_2|(2\pi)^6}\delta^4(q_1+q_2+Q)\text{Tr}q_1\cdot\gamma\gamma_\lambda(1-\gamma_5)q_2\cdot\gamma\gamma_\kappa(1-\gamma_5) = N[Q_\lambda Q_\kappa - \eta_{\kappa\lambda}Q^2] \tag{6}$$

where $Q = p_e - p_\mu$. Prove this formula, find N, and evaluate $|\mathcal{M}|^2$ for a *polarized* μ, integrated over the neutrino phase space and summed over the spin of the electron in the final state. Evaluate the diffential rate $d^2\Gamma/(dE_e d\Omega)$, where E_e is the electron energy in the muon's rest frame.

c) Explain what feature of this distribution implies parity violation. How does the distribution change for the charge conjugated process $\mu^+ \to e^+ + \bar{\nu}_\mu + \nu_e$?

d) Calculate the total rate assuming $m_e = 0$. By comparing this to the observed rate, find an approximate value for G_F implied by this tree approximation. How does it compare to the current best value?