

Standard Model/Quantum Field Theory III

Problem Set 1

Due: Wednesday, 22 January 2020

Suggested reading: QFT Notes, Ch 25; Sr, Secs 75-79; P, Ch 19; Sc, Ch 30. Here Sr=Srednicki, P=Peskin&Schroeder, and Sc=Schwartz. These sources cover the same material, but from different points of view. My notes are self-contained.

1. In our discussion of the axial current anomaly, the triangle diagrams with an axial current at one vertex and vector currents at the other two vertices played a central role. In momentum space this diagram must be proportional to a 3-index pseudotensor $X^{\mu\rho\sigma}(k_1, k_2)$ constructed in terms of the two independent vector momenta k_1^μ, k_2^μ and the Levi-Civita tensor. $\epsilon^{\mu\nu\rho\sigma}$. We found two candidates for X which satisfied Bose statistics in the vector vertices $X^{\mu\rho\sigma}(k_1, k_2) = X^{\mu\sigma\rho}(k_2, k_1)$ and current conservation of the vector currents $k_{1\rho}X^{\mu\rho\sigma} = k_{2\sigma}X^{\mu\rho\sigma} = 0$. Prove that these two candidates are in fact equal to each other. [You may assume that the vector momenta are on photon shell $k_1^2 = k_2^2 = 0$. The offshell case is given in a footnote of my notes.]:

$$\epsilon^{\mu\rho\sigma\tau}(k_1 - k_2)_\tau + \frac{2(k_1 + k_2)^\rho k_{1\tau} k_{2\lambda} \epsilon^{\mu\sigma\lambda\tau}}{(k_1 + k_2)^2} - \frac{2(k_1 + k_2)^\sigma k_{1\tau} k_{2\lambda} \epsilon^{\mu\rho\lambda\tau}}{(k_1 + k_2)^2} = + \frac{2(k_1 + k_2)^\mu k_{1\tau} k_{2\lambda} \epsilon^{\rho\sigma\lambda\tau}}{(k_1 + k_2)^2}. \quad (1)$$

A pedestrian approach to the proof is to work in the rest frame of the two photons $\mathbf{k}_1 = -\mathbf{k}_2$, in which case you have to go through all the distinct choices of $\mu\rho\sigma$: all space; two space, one time; 1 space, two times; all times. Some of these cases are trivial, but explain why. If you can find a frame independent proof of this identity, by all means present it!

2. Decay rate for $\pi_0 \rightarrow \gamma\gamma$.

a) Define F_{π_0} by

$$\langle 0 | j_{35}^\mu(0) | \mathbf{q}, \pi_0 \rangle \equiv \frac{i F_{\pi_0} q^\mu}{(2\pi)^{3/2} \sqrt{2\omega_{\pi_0}(\mathbf{q})}},$$

where $j_{35}^\mu = \bar{q} \frac{\tau_3}{2} \gamma_5 \gamma^\mu q$ is the 3 component of the axial isospin current. Since isospin is a very good symmetry of the strong interactions, we may approximate $F_{\pi_0} \approx F_{\pi^-} \approx 93$ MeV. Consulting the reduction formalism developed last semester, confirm the assertion in class, that the matrix element of j_{35}^μ between the vacuum and a two photon state behaves for $(k_1 + k_2)^2 \rightarrow -m_{\pi_0}^2$ as

$$\langle \mathbf{k}_1 \epsilon_1, \mathbf{k}_2 \epsilon_2 | j_{35}^\mu(0) | 0 \rangle \sim \frac{(k_1 + k_2)^\mu}{(k_1 + k_2)^2 + m_{\pi_0}^2} \frac{F_{\pi_0} \mathcal{M}_{\pi_0 \rightarrow \gamma\gamma}}{(2\pi)^3 \sqrt{4|\mathbf{k}_1||\mathbf{k}_2|}}, \quad \text{for } (k_1 + k_2)^2 \rightarrow -m_{\pi_0}^2$$

where \mathcal{M} is the Feynman amplitude for the decay process. (Of course, since the pion is unstable, there is not really a pole for a real value of $(k_1 + k_2)^2$. In this case the reduction formula makes sense only if the time T is taken to satisfy $T \ll \tau_\pi$ where τ_π is the pion

lifetime. In fact the pole should move into the complex plane by an amount $-im_{\pi_0}\Gamma_{\pi_0}$, with Γ_{π_0} the rate for π_0 decay in its rest frame. Below you will find that $\Gamma_{\pi_0} = O(\alpha^2 m_{\pi_0}) \ll m_{\pi_0}$, so the pole is extremely close to the real axis!)

- b) As we have seen in class, the triangle anomaly implies that, in the limit of massless quarks, the matrix element of j_{35}^μ has a pole at $(k_1 + k_2)^2 = 0$. We can interpret this by saying that the mass of the pion vanishes with massless up and down quarks, in which case the pole obtained in part a) just becomes the massless pole required by the anomaly. To use the anomaly to estimate π_0 decay, we must suppose that giving the up and down quarks their relatively small masses moves the pole away from zero, giving the pion its small mass, but has a very small effect on the residue of the pole. In this case, \mathcal{M} can be approximated by comparison with the residue of the massless pole required by the anomaly. Read off the resulting prediction for \mathcal{M} and use it to approximately calculate the rate for $\pi_0 \rightarrow \gamma\gamma$. A clear way to express the approximation is as the tree approximation to $\pi_0 \rightarrow \gamma\gamma$ using the effective Lagrangian

$$\mathcal{L}_{eff} = -\frac{1}{2}((\partial\pi_0)^2 + m_\pi^2\pi_0^2) + \frac{\alpha}{8\pi F_{\pi_0}}\pi_0\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \quad (2)$$

where m_π is the actual mass of the π_0 .

- c) Look up the experimental lifetime τ_{π_0} for the π_0 . Since two photons are almost always the final state, $1/\tau_{\pi_0}$ is a good approximation to the rate for decay into $\gamma\gamma$. Compare the experimental result to the calculation of parts a) and b), and discuss the accuracy of the approximation.

3. The last problem on the take home exam is important enough to justify giving it a careful second look. Rework your solution correcting any mistakes and filling in all important details. I reproduce the problem below:

From Final: In class we used a nonrenormalizable effective field theory model to obtain low energy pion scattering from spontaneously broken chiral symmetry. Repeat the calculation, using instead the linear σ model with the renormalizable Lagrangian:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi^a)^2 - \frac{\lambda}{4}(\phi^a\phi^a - F_\pi^2)^2$$

with ϕ^a a 4-vector field under chiral $O(4)$ which is isomorphic to $SU(2) \times SU(2)$.

- a) (7 points) After shifting the field by its classical expectation value, write out the Lagrangian in terms of three pion fields π_a and one other scalar field σ . Show that the particle content is three massless Goldstone bosons (pions) and a massive scalar field, σ , which is the component of ϕ^a that gets an expectation value.
- b) (7 points) Draw and evaluate all of the 4 point tree diagrams contributing to $\pi\pi$ scattering and to $\pi\sigma$ scattering, using the Cartesian basis for pion fields π_a , $a = 1, 2, 3$, Then changing to the charge basis $\pi_\pm = (\pi_1 \pm i\pi_2)/\sqrt{2}$, $\pi_0 = \pi_3$, obtain the Feynman amplitudes for the scattering processes $\pi^+\pi^+ \rightarrow \pi^+\pi^+$, $\pi^+\pi^0 \rightarrow \pi^+\pi^0$, $\pi^+\pi^- \rightarrow \pi^+\pi^-$, $\pi^+\pi^- \rightarrow \pi^0\pi^0$, and $\pi^0\sigma \rightarrow \pi^0\sigma$.
- c) (6 points) Calculate the center of mass differential cross section for each process. Obtain the low energy behavior of the cross section in each case.

- d) (7 points) Now add a symmetry breaking term $-c\phi^4$, an isoscalar. What does “vacuum alignment” tell you about the VEV you must use for ϕ^a ? Find the mass of the pions in the presence of this symmetry breaking.
- e) (7 points) Evaluate the amplitudes for all the processes of b) in the presence of symmetry breaking. Find the threshold limit of each amplitude.
- f) (6 points) Compare your answers to those we got in class, taking into account the new physics associated with the σ particle in this model, which played no role in the class discussion (why?).