Standard Model/Quantun Field Theory
Solution Set 1

Due: 6 September 2019

Suggested reading: QFT Notes Chs 16-18; Textbook: Secs. 58-59,61.

1. **Gaussian Integrals**

a) Let x_k, $k = 1, \ldots, N$ be a set of real integration variables, and let M_{kl} be a real symmetric $N \times N$ matrix. Calculate the integral

$$\int_{-\infty}^{\infty} \prod_k dx_k e^{-(1/2)x_k M_{kl} x_l + iJ_k x_k} \propto (\det M)^{-1/2} e^{(1/2)(iJ_k)(M^{-1})_{kl}(iJ_l)} \quad (1)$$

where the "source" J_k is a set of N real constants. It is understood that repeated indices are summed.

Solution: First calculate the source dependence by completing the square: change variables to $x_k = y_k + c_k$

$$-(1/2)(y_k + c_k)M_{kl}(y_l + c_l) + iJ_k(y_k + c_k) = -(1/2)y_k M_{kl} y_l - (1/2)c_k M_{kl} c_l + iJ_k c_k \quad (2)$$

if $c = iM^{-1}J$. Then

$$-(1/2)c_k M_{kl} c_l + iJ_k c_k = \frac{i}{2} J_k c_k = \frac{1}{2} iJ_k (M^{-1})_{kl} iJ_l \quad (3)$$

as desired. To do the Gaussian integral over y_k change coordinates with an orthogonal matrix which diagonalizes the real symmetric matrix M. Let the eigenvalues be μ_k. Then we just have N independent Gaussian integrals the value of each is $\sqrt{2\pi/\mu_k}$ so the result is

$$\int_{-\infty}^{\infty} \prod_k dy_k e^{-(1/2)y_k M_{kl} y_l} = \frac{(2\pi)^{N/2}}{\sqrt{\prod_k \mu_k}} = \frac{(2\pi)^{N/2}}{\sqrt{\det M}} \quad (4)$$

b) Extend this result to integration over Grassmann numbers a_k and $\bar{a}_k$ with source η_k and $\bar{\eta}_k$:

$$\int \prod_k da_k d\bar{a}_k e^{-\bar{a}_k M_{kl} a_l + i\bar{\eta}_k a_k + i\bar{a}_k \eta_k} \propto (\det M) e^{(i\bar{\eta}_k)(M^{-1})_{kl}(i\eta_l)} \quad (5)$$

Here a_k and $\bar{a}_k$ are independent Grassmann variables. See notes chapter 13 for a very brief intro to Grassmann integration.

Solution: Checking the source dependence follows the same steps as part a). To do the Gaussian integral, use the properties of Grassmann numbers to expand the exponential: only the Nth term survives integration.

$$\int \prod_j da_j d\bar{a}_j e^{-\bar{a}_k M_{kl} a_l} = \frac{(-)^N}{N!} \int \prod_j da_j d\bar{a}_j (\bar{a}_k M_{kl} a_l)^N = \det M \quad (6)$$

by Eq. (13.4) in the Lecture notes.

2. Find the formula, for arbitrary spacetime dimension D, for the degree of divergence $\mathcal{D}$ for a general diagram in:

a) Scalar electrodynamics

$$\mathcal{L}^0_{\rm QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (\partial + iQ_0 A)\phi^\dagger \cdot (\partial - iQ_0 A)\phi - \mu^2\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2 \tag{7}$$

Solution: We start with the general formula

$$\begin{aligned}\mathcal{D} &= DL - 2I_B + \sum_n nV_n = D(I_B - V_0 - V_1 + 1) - 2I_B - I_F + V_1 \\ &= D + (D-2)I_B - DV_0 + (1-D)V_1 \end{aligned} \tag{8}$$

For scalar QED, each V_0 has 4 ends, and each V_1 has 3 ends, so $4V_0 + 3V_1 = 2I_B + E_B$ and

$$\begin{aligned}\mathcal{D} &= D + \frac{D-2}{2}(4V_0 + 3V_1 - E_B) - DV_0 + (1-D)V_1 \\ &= D - \frac{D-2}{2}E_B + \frac{D-4}{2}(2V_0 + V_1) \end{aligned} \tag{9}$$

For $D = 4$, this reduces to $\mathcal{D} = 4 - E_B$. The superficially divergent graphs have $E_B = 0, 1, 2, 3, 4$, with quartic, cubic, quadratic, linear, and log divergences respectively.. The $E_B = 0$ diagrams are vacuum bubbles which cancel in the functional averages. The 1PIR $E_b = 1$ case is the photon tadpole diagram, which is zero by the requirement that the current have zero vacuum expectation value. (Charge conservation forbids a scalar tadpole.) The 1PIR $E_B = 2$ diagrams are either the photon or scalar self energy. There are two of each at 1 loop. In addition the scalar self energy has an additional diagram with the ϕ^4 vertex. Because of charge conservation demands that the number of scalar external lines is even and charge conjugation invariance demands that the three photon amplitude is zero, the 1PIR $E_B = 3$ diagrams must have two scalar and 1 photon external lines. At 1 loop there are 4 possibe such diagrams. Finally the $E_B = 4$ diagrams are four scalars, two scalars and two photons (Compton scattering), and 4 photons (scattering of light by light)..because the external lines can be permuted there are many diagrams, some for each process are shown in the attached pages.

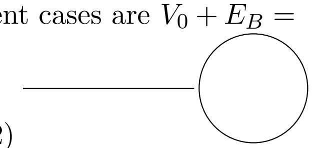
b) Hermitian scalar ϕ^3 theory:

$$\mathcal{L} = -\frac{1}{2}((\partial\phi)^2 + \mu^2\phi^2) - \frac{g}{3!}\phi^3 \tag{10}$$

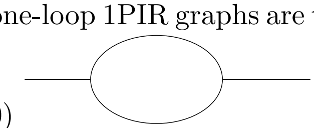
Solution: In this case there is only one vertex with three ends and no derivatives: $3V_0 = 2I_B + E_B$

$$\mathcal{D} = D + (D-2)I_B - DV_0 = D + \frac{D-2}{2}(3V_0 - E_B) - DV_0 = D + \frac{D-6}{2}V_0 - \frac{D-2}{2}E_B$$

For $D = 4$ this reduces to $\mathcal{D} = 4 - V_0 - E_B$. The superficially divergent cases are $V_0 + E_B = 0, 1, 2, 3, 4$. The divergent one-loop 1PIR graphs are the tadpole ($\mathcal{D} = 2$)



and the self energy ($\mathcal{D} = 0$)



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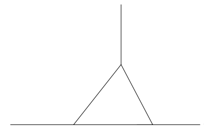
For $D=6$ the V_0 term drops out and $\mathcal{D} = 6 - 2E_B$. The superficially divergent cases are $E_B = 0, 1, 2, 3$. The divergent one-loop 1PIR graphs are the tadpole ($\mathcal{D}=4$), the self energy ($\mathcal{D}=2$), and the 3 scalar triangle ($\mathcal{D}=0$) .

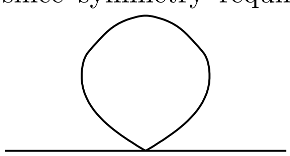
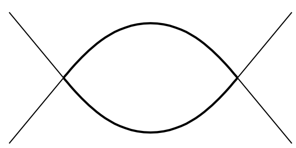
c) Hermitian scalar ϕ^4 theory:

$$\mathcal{L} \quad = \quad -\frac{1}{2}((\partial\phi)^2 + \mu^2\phi^2) - \frac{g}{4!}\phi^4 \tag{11}$$

Solution: In this case $4V_0 = 2I_B + E_B$, and

$$\mathcal{D} \quad = \quad D + (D-2)I_B - DV_0 = D + \frac{D-2}{2}(4V_0 - E_B) - DV_0 = D + (D-4)V_0 - \frac{D-2}{2}E_B$$

For $D = 4$ this is $\mathcal{D} = 4 - E_B$, so the superficially divergent cases are $E_B = 0, 2, 4$, since symmetry requires E_B is even. For $E_B = 2, 4$ the the 1 loop 1PIR diagrams are

 and  respectively

In each case, draw all the one loop diagrams which are UV divergent in 4 spacetime dimensions. Do the same for the ϕ^3 case in 6 spacetime dimensions.

3. Calculate the differential cross section for

a) The elastic scattering process $e^- + \mu^- \rightarrow e^- + \mu^-$ in lowest order perturbation theory in QED, assuming the initial particles are unpolorized and that the spin of final particles is not observed. You may work in the center of mass system.

Solution: Since the electron and muon are distiguishable there is only one tree diagram for this process. Applying the rules gives the amplitude

$$\begin{aligned}\mathcal{M} \quad &= \quad (ie)^2\bar{u}_e\gamma^\mu u_e \bar{u}_\mu \gamma^\mu u_\mu \frac{-i}{(p_e' - p_e)^2} \\ \frac{1}{4}\sum_{spins}|\mathcal{M}|^2 \quad &= \quad e^4\frac{\mathrm{Tr}[\gamma^\mu(m_e - p_e\cdot\gamma)\gamma^\nu(m_e - p_e'\cdot\gamma)]\mathrm{Tr}[\gamma_\mu(m_\mu - p_\mu\cdot\gamma)\gamma_\nu(m_\mu - p_\mu'\cdot\gamma)]}{4(p_e'-p_e)^4}\end{aligned}$$

Evaluation of a trace factor:

$$\begin{aligned}\mathrm{Tr}[\gamma^\mu(m_e - p_e\cdot\gamma)\gamma^\nu(m_e - p_e'\cdot\gamma)] \quad &= \quad -4m_e^2\eta^{\mu\nu} + 4[p_e^\mu p_e^{\nu\prime} + p_e^\nu p_e^{\mu\prime} - \eta^{\mu\nu}p_e\cdot p_e'] \\ &= \quad -4(m_e^2 + p_e\cdot p_e')\eta^{\mu\nu} + 4[p_e^\mu p_e^{\nu\prime} + p_e^\nu p_e^{\mu\prime}] \end{aligned}\tag{12}$$

with the other trace obtained by $p_e \rightarrow p_\mu$. Note that $2(m_e^2 + p_e\cdot p_e') = -(p_e' - p_e)^2 = -(p_\mu' - p_\mu)^2 = 2(m_\mu^2 + p_\mu\cdot p_\mu') \equiv -q^2$. The product of the two traces in the spin averaged squared amplitude is then:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{q^4}[4q^4+4q^2(p_e\cdot p'_e+p_\mu\cdot p'_\mu)+8(p_e\cdot p_\mu p'_e\cdot p'_\mu+p_e\cdot p'_\mu p'_e\cdot p_\mu)]$$
$$= \frac{e^4}{q^4}[-4q^2(m_e^2+m_\mu^2)+8(p_e\cdot p_\mu p'_e\cdot p'_\mu+p_e\cdot p'_\mu p'_e\cdot p_\mu)] \quad (13)$$

In the center of mass $p_\mu\cdot p_e = p'_\mu\cdot p'_e = -\boldsymbol{p}_e^2 - E_\mu E_e$, $p'_\mu\cdot p_e = p_\mu\cdot p'_e = -\boldsymbol{p}_e^2\cos\theta - E_\mu E_e$ and $q^2 = 2\boldsymbol{p}_e^2(1-\cos\theta) = 4\boldsymbol{p}_e^2\sin^2(\theta/2)$. So

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{q^4}[-4q^2(m_e^2+m_\mu^2)+8((p_e^2\cos\theta+E_\mu E_e)^2+(p_e^2\cos\theta-E_\mu E_e)^2]$$
$$= \frac{e^4}{q^4}[-4q^2(m_e^2+m_\mu^2)+16(p_e^4\cos^2\theta+(p_e^2+m_\mu^2)(p_e^2+m_e^2)] \quad (14)$$

The differential cross section is

$$d\sigma = \frac{d^3p'_e d^3p'_\mu}{(2\pi)^6 16E'_e E'_\mu E_e E_\mu}(2\pi)^4\delta(p'_e+p'_\mu-p_e-p_\mu)\frac{|\mathcal{M}|^2}{v} \quad (15)$$

In the C of M $v = |\boldsymbol{p}_e|(E_e+E_\mu)/(E_e E_\mu)$. Using the spatial delta function to deo the p'_μ integal and simplitying gives

$$\frac{d\sigma}{d\Omega} = \frac{p_e'^2 dp'_e}{(2\pi)^2 16E'_e E'_\mu}\delta(E'_e+E'_\mu-E_e-E_\mu)\frac{|\mathcal{M}|^2}{p_e(E_e+E_\mu)}$$
$$= \frac{1}{64\pi^2}\frac{|\mathcal{M}|^2}{(E_e+E_\mu)^2}$$
$$= \frac{e^4}{64\pi^2 q^4 s}[-4q^2(m_e^2+m_\mu^2)+16(p_e^4\cos^2\theta+(p_e^2+m_\mu^2)(p_e^2+m_e^2)]$$
$$= \frac{\alpha^2}{q^4 s}[-q^2(m_e^2+m_\mu^2)+4(p_e^4\cos^2\theta+(p_e^2+m_\mu^2)(p_e^2+m_e^2)] \quad (16)$$

Here the p'_e integral was done using the delta function which sets $p'_e = p_e$ and divides by $p_e(E_e+E_\mu)/(E_e E_\mu)$

b) The inelastic scattering process $e^- + e^+ \to \mu^- + \mu^+$ in lowest order perturbation theory in QED, assuming the initial particles are unpolorized and that the spin of final particles is not observed. You may work in the center of mass system.

Solution: The amplitude for this process is related to that for part a). The rules say

$$\mathcal{M} = (ie)^2\bar{v}_e\gamma^\mu u_e\bar{u}_\mu\gamma^\mu v_\mu\frac{-i}{(p_{e+}+p_e)^2}$$
$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = e^4\frac{\mathrm{Tr}[\gamma^\mu(m_e-p_e\cdot\gamma)\gamma^\nu(-m_e-p_{e+}\cdot\gamma)]\mathrm{Tr}[\gamma_\mu(-m_\mu-p'_{\mu+}\cdot\gamma)\gamma_\nu(m_\mu-p'_\mu\cdot\gamma)]}{4(p_{e+}+p_e)^4}$$

Comparing the two expressions for the squared amplitudes, we see that we get from a) to b) by the sustitutions $p'_e \to -p_{e+}$ and $p_\mu \to p'_{\mu+}$. Thus we can immediately write

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{s^2}[4s(m_e^2+m_\mu^2)+8(p_e\cdot p'_{\mu+}p_{e+}\cdot p'_\mu + p_e\cdot p'_\mu p_{e+}\cdot p'_{\mu+})] \tag{17}$$

where we used $q^2 \to (p_e+p_{e+})^2 \equiv -s = -4E_e^2$. The differential cross section for this process is

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{p'_\mu}{64\pi^2 p_e s}\frac{1}{4}\sum|\mathcal{M}|^2 \\ &= \frac{\alpha^2\sqrt{1-4m_\mu^2/s}}{s\sqrt{1-4m_e^2/s}}\left[\frac{m_e^2+m_\mu^2}{s}+\frac{2}{s^2}(p_e\cdot p'_{\mu+}p_{e+}\cdot p'_\mu + p_e\cdot p'_\mu p_{e+}\cdot p'_{\mu+})\right] \end{aligned}\tag{18}$$

Of course this process requires $s \geq 4m_\mu^2$.

4. The π^0 is a pseudoscalar strongly interacting particle. We shall see later that its coupling to the electromagnetic field can be described by a term in the Lagrangian of the form

$$G\phi_\pi \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{19}$$

where ϕ_π is a hermitian pseudoscalar field and F is the electromagnetic field strength tensor. Calculate in the pion rest frame the differential and total rates for the decay process $\pi_0 \to 2$ photons.

Solution: The vertex from this term in the Lagrangian gives the lowest order amplitude

$$\mathcal{M} = 2iG\epsilon^{\mu\nu\rho\sigma}2iq^1_\mu\epsilon^*_{1\nu}2iq^2_\mu\epsilon^*_{2\nu} \tag{20}$$

Specializing to the center of mass, $\boldsymbol{q}_2 = -\boldsymbol{q}_1$, $q_2^0 = q_1^0 = |\boldsymbol{q}_1|$, and choosing a gauge where $\epsilon_1^0 = \epsilon_2^0 = 0$, leads to

$$\begin{aligned}\mathcal{M} &= -8iG\epsilon^{0klm}(q_0^1 q_l^2 - q_l^1 q_0^2)\epsilon^*_{1k}\epsilon^*_{2m} = +16iG|\boldsymbol{q}_1|\epsilon^{klm}\epsilon^*_{1k}q_l^1\epsilon^*_{2m} \\ &= -16iG|\boldsymbol{q}_1|\boldsymbol{q}_1\cdot(\boldsymbol{\epsilon}_1^*\times\boldsymbol{\epsilon}_2^*) \\ \sum_{pol}|\mathcal{M}|^2 &= 256G^2\boldsymbol{q}_1^2\epsilon^{klm}q_l^1\epsilon^{prs}\delta_{kp}q_r^1\delta_{ms} = 512G^2\boldsymbol{q}_1^4 \end{aligned}\tag{21}$$

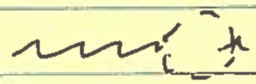
The formula for the rate in the pion rest frame, remembering that the two photons in the final state are indistinguishable, is

$$\begin{aligned}\Gamma &= \frac{1}{2}\int\frac{d^3q_1}{(2\pi)^6 4\boldsymbol{q}_1^2}\frac{1}{2m_\pi}(2\pi)^4\delta(m_\pi - 2|\boldsymbol{q}_1|)512G^2\boldsymbol{q}_1^4 \\ &= \frac{1}{16\pi}\frac{1}{2m_\pi}512G^2\frac{m_\pi^4}{16} = \frac{m_\pi^3 G^2}{\pi}\end{aligned}\tag{22}$$

Feynman Diagrams for Problem 4 a).

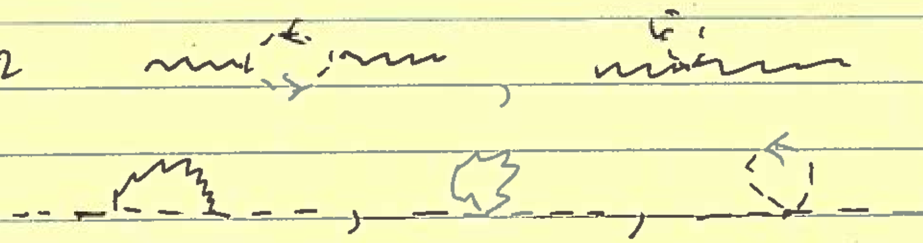
~~Diagram~~ $\mathcal{D} \geq 0$ 1PIR diagrams

$\mathcal{D} = 3$ $E_B = 1$

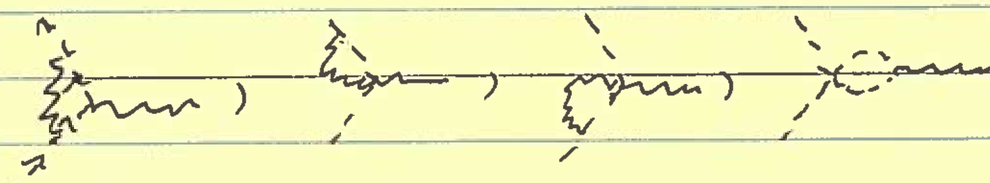


$= 0$ (By $\langle 0 | j^\mu(x) | 0 \rangle = 0$

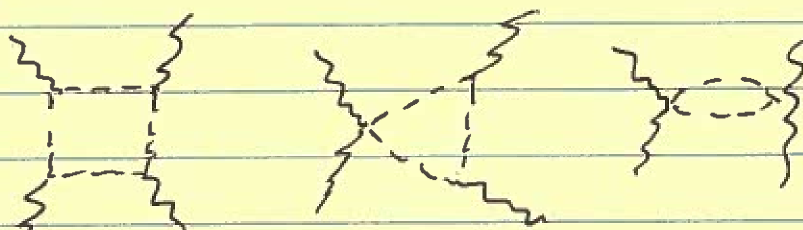
$\mathcal{D} = 2$ $E_B = 2$



$\mathcal{D} = 1$ $E_B = 3$



$\mathcal{D} = 0$ $E_B = 4$



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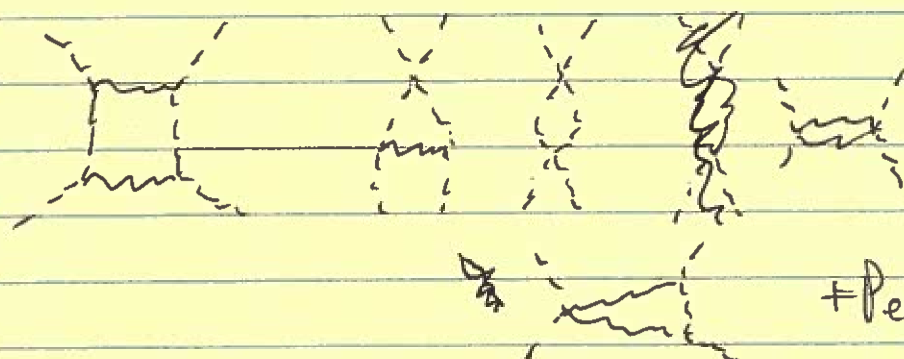

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