Standard Model/Quantun Field Theory

Problem Set 2

Due: Monday, 23 September 2019

Suggested reading: QFT Notes Chs 18-19; Textbook: Secs. 62-65.

5. As discussed in class, gauge invariance in QED implies the Ward identity

$$q_\mu \Gamma^\mu(p', p) = m + \gamma \cdot p' + \Sigma(p') - (m + \gamma \cdot p + \Sigma(p)) \tag{1}$$

where $q = p' - p$. Sandwiching both sides with on **physical** mass shell spinors reduces this statement to $q_\mu \bar{u}' \Gamma^\mu u = 0$.

a) Show that the quantities

$$\bar{u}'\gamma^\mu u, \quad \bar{u}'(p'+p)^\mu u, \quad \bar{u}'\sigma^{\mu\nu} q_\nu u$$

satisfy this conservation law (recall that $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$).

Solution:

$$q_\mu \bar{u}'\gamma^\mu u = \bar{u}'(\gamma \cdot p' + m - \gamma \cdot p - m)/u = 0 \tag{2}$$

by the free Dirac equation.

$$q_\mu \bar{u}'(p'+p)^\mu u = \bar{u}'(p'^2 - p^2)u = 0 \tag{3}$$

since $p'^2 = p^2 = -m^2$.

$$q_\mu \bar{u}'\sigma^{\mu\nu} q_\nu u = 0 \tag{4}$$

because $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$.

b) Prove the identity

$$2i\bar{u}'\sigma^{\mu\nu} q_\nu u = 2\bar{u}'(p'+p)^\mu u - 4m\bar{u}'\gamma^\mu u.$$

And argue that the most general Lorentz and parity covariant, and gauge invariant form for Γ^μ, when sandwiched between on physical mass shell spinors, can be taken to be

$$\Gamma^\mu = \gamma^\mu F_1(q^2) - \frac{i}{2m}\sigma^{\mu\nu} q_\nu F_2(q^2).$$

Solution:

$$\begin{aligned} 2i\bar{u}'\sigma^{\mu\nu} q_\nu u &= -\bar{u}'(\gamma^\mu q \cdot \gamma - q \cdot \gamma \gamma^\mu)u = -\bar{u}'(\gamma^\mu(p' \cdot \gamma + m) + (p \cdot \gamma + m)\gamma^\mu)u \\ &= -\bar{u}'(-2p'^\mu + 2m\gamma^\mu - 2p^\mu + 2m\gamma^\mu)u = 2\bar{u}'(p'+p)^\mu u - 4m\bar{u}'\gamma^\mu u \end{aligned} \tag{5}$$

as desired. Lorentz covariance requires that we form a polar four vector out of the 2 momenta and the γ matrices. The three four vectors listed in part a are each conserved, but the identity proved here shows only two are independent. Other four vector structures that can be formed, such as replacing q by $p+p'$ in the third one or $p+p'$ by q in the second one are not conserved. Including a factor of γ_5 makes it an axial vector violating parity. So any two of the structures in part a) can be chosen. They can each be multiplied by functions of the scalars formed from the momenta. But $p^2 = p'^2 = -m^2$ are fixed leaving only q^2. So the form quoted here is the most general one.

6. In class we calculated the proper vertex function in terms of two Feynman parameters:

$$\bar{u}'\Gamma^\mu(p',p)u \equiv \bar{u}'\left[\gamma^\mu F_1(q^2) + \frac{[\gamma^\mu, q\cdot\gamma]}{4m}F_2(q^2)\right]u \tag{6}$$

$$\begin{aligned} F_1(q^2) &= 1 + \frac{Q_0^2}{8\pi^2}\int dxdy\Bigg(\ln\frac{\Lambda^2}{\lambda^2(1-x-y)+m^2(x+y)^2+xyq^2} - \frac{3}{2} \\ &\quad + \frac{m^2((x+y)^2 - 2(1-x-y) - q^2(1-x)(1-y))}{\lambda^2(1-x-y)+m^2(x+y)^2+xyq^2}\Bigg) \end{aligned} \tag{7}$$

$$F_2(q^2) = \frac{Q_0^2}{4\pi^2}\int dxdy\frac{m^2(x+y)(1-x-y)}{\lambda^2(1-x-y)+m^2(x+y)^2+xyq^2} \tag{8}$$

where λ is a small photon mass introduced as an infrared cutoff. Also the range of integration is $x+y\leq 1$ In this problem we study the limit $\lambda\to 0$.

a) First make a convenient change of variables $x = u(1-v)$, $y = uv$. show that the range of integration for the new variables is $0 < u, v < 1$ and that

$$\begin{aligned} F_1(q^2) &= 1 + \frac{Q_0^2}{8\pi^2}\int dvudu\Bigg(\ln\frac{\Lambda^2}{\lambda^2(1-u)+u^2(m^2+v(1-v)q^2)} - \frac{3}{2} \\ &\quad + \frac{m^2(u^2-2(1-u)) - q^2(1-u+u^2v(1-v))}{\lambda^2(1-u)+m^2u^2+q^2u^2v(1-v)}\Bigg) \end{aligned} \tag{9}$$

$$F_2(q^2) = \frac{Q_0^2}{4\pi^2}\int dvudu\frac{m^2u(1-u)}{\lambda^2(1-u)+u^2(m^2+v(1-v)q^2)} \tag{10}$$

Solution: The range of x, y is $0 < x < 1, 0 < y < 1-x$, which translates to $0 < u(1-v) < 1, 0 < uv < 1-u+uv$ which means that $u, v, (1-v)$ have the same sign and $u < 1$. v and $1-v$ can have the same sign only it $0 < v < 1$, so it follows that $0 < u < 1$. The Jacobian of the variable change is $(1-v)u + uv = u$. Then substitution for x, y establishes the result.

b) Show that one may safely set $\lambda = 0$ in the integrand of F_2 and in all the terms in the integrand of F_1, except for the terms in the numerator of the last term which have no factors of u. For all of the terms allowing $\lambda = 0$, the u integral is elementary so do each one!

Solution: In F_2 the u^2 in the numerator removes the potential IR divergence from u near zero, so setting $\lambda = 0$ results in

$$F_2(q^2) = \frac{Q_0^2}{4\pi^2}\int dvdu\frac{m^2(1-u)}{(m^2+v(1-v)q^2)} = \frac{Q_0^2}{8\pi^2}\int dv\frac{m^2}{(m^2+v(1-v)q^2)} \tag{11}$$

Call the part of F_1 not evaluated in part c) below $\hat{F}_1$ and set $\lambda = 0$ in it

$$\begin{aligned}\hat{F}_1(q^2) &\to 1 + \frac{Q_0^2}{8\pi^2}\int dvdu\Big(u\ln\frac{\Lambda^2}{(m^2+v(1-v)q^2)} - \frac{3}{2}u - 2u\ln u \\ &\quad + \frac{m^2(u+2) - q^2(-1+uv(1-v))}{m^2+q^2v(1-v)}\Big) \\ &\to 1 + \frac{Q_0^2}{8\pi^2}\int dv\Big(\frac{1}{2}\ln\frac{\Lambda^2}{(m^2+v(1-v)q^2)} - \frac{1}{4} \\ &\quad + \frac{5m^2/2 + q^2(1-v(1-v)/2)}{m^2+q^2v(1-v)}\Big) \end{aligned} \tag{12}$$

c) This leaves all the λ dependence in the integral

$$\int dvudu\frac{-2m^2-q^2}{\lambda^2(1-u)+u^2(m^2+q^2v(1-v))}; \tag{13}$$

which would be log divergence with $\lambda = 0$, so we expect there to be a $\ln\lambda$ dependence. One way to extract it is to break the u integration range into two regions: Pick an ϵ satisfying $\lambda/m \ll \epsilon \ll 1$. For $0 < u < \epsilon$ show that it is valid to replace the denominator by $\lambda^2 + u^2(m^2+q^2v(1-v))$ and then do the u integral. For $\epsilon < u < 1$ set $\lambda = 0$ (why valid?) and do the integral. Show that the dependence on ϵ cancels between the two terms.

Solution: For $0 < u < \epsilon$, u is musch smaller than 1, so the simplification is valid. Then

$$\begin{aligned}\int_0^\epsilon udu\frac{1}{\lambda^2+u^2(m^2+q^2v(1-v))} &= \frac{1}{2}\ln\frac{\lambda^2+\epsilon^2(m^2+q^2v(1-v))}{\lambda^2} \\ &= \frac{\ln(\epsilon/\lambda) + (1/2)\ln(m^2+q^2v(1-v))}{m^2+q^2v(1-v)} + O\frac{\lambda^2}{m^2\epsilon^2}\end{aligned} \tag{14}$$

For $\epsilon < u < 1$ we can set $\lambda = 0$ because $\lambda/m \ll \epsilon$ and get

$$\int_\epsilon^1 \frac{du}{u}\frac{1}{m^2+q^2v(1-v))} = \frac{-\ln\epsilon}{m^2+q^2v(1-v))} \tag{15}$$

Combining the two regions gives

$$\int dvudu\frac{-2m^2-q^2}{\lambda^2(1-u)+u^2(m^2+q^2v(1-v))} \sim \frac{1}{2}\int dv\frac{-2m^2-q^2}{m^2+q^2v(1-v))}\ln\frac{m^2+q^2v(1-v))}{\lambda^2} \tag{16}$$

d) Put everything together expressing F_1 and F_2 as single integrals over v, showing the explicit $\ln\Lambda$ and $\ln\lambda$ terms. The final result is accurate up to terms that vanish as $\lambda \to 0$.

Solution: F_2 is given in part b). Combining the result of part c) with the work on F_1 done in part b)

$$\begin{aligned}F_1(q^2) &= 1 + \frac{Q_0^2}{8\pi^2}\int dv\Big(\frac{1}{2}\ln\frac{\Lambda^2}{(m^2+v(1-v)q^2)} - \frac{1}{4} \\ &\quad + \frac{5m^2/2+q^2(1-v(1-v)/2)}{m^2+q^2v(1-v)} - \frac{1}{2}\frac{2m^2+q^2}{m^2+q^2v(1-v))}\ln\frac{m^2+q^2v(1-v))}{\lambda^2}\Big)\end{aligned} \tag{17}$$

7. **Bremsstrahlung.** We want to calculate, in tree approximation. the cross-section for scattering of an electron in a static external field with emission of a photon, in particular a soft photon. This problem steps you through calculations sketched in class filling in some missing details.

a) In lowest order in the external field there are two graphs. Write down the amplitude for the scattering of an electron of 4-momentum p_1 to a final state p_2 with emission of a photon of momentum k and polarization ϵ in a static field $A^\mu_{\rm ext}(\mathbf{x})$.

Solution:

$$\begin{aligned}\mathcal{M} &= \bar{u}(p_2)\Big\{ie\gamma\cdot\epsilon^*(k)\frac{-i}{m+\gamma\cdot(p_2+k)}ie\gamma\cdot\tilde{A}(p_2+k-p_1) &(18)\\ &+ie\gamma\cdot\tilde{A}(p_2-p_1+k)\frac{-i}{m+\gamma\cdot(p_1-k)}ie\gamma\cdot\epsilon^*(k)\Big\}u(p_1) &(19)\\ &= ie^2\bar{u}(p_2)\Big\{\frac{2p_2\cdot\epsilon^*-\gamma\cdot\epsilon^*(k)\gamma\cdot k}{2p_2\cdot k}\gamma\cdot\tilde{A}(p_2+k-p_1) &(20)\\ &+\gamma\cdot\tilde{A}(p_2-p_1+k)\frac{2p_1\cdot\epsilon^*+\gamma\cdot k\gamma\cdot\epsilon^*(k)}{-2p_1\cdot k}\Big\}u(p_1) &(21)\end{aligned}$$

For a static field, $\tilde{A}_\mu(q)=2\pi\delta(q^0)\tilde{A}_\mu(\vec{q})$, so we write $\mathcal{M}=2\pi\delta(q^0)M$.

b) Take as variables the initial momentum $\mathbf{p_1}$, the magnitude and direction of photon momentum k, $\hat{\mathbf{k}}$, and the direction of the final electron momentum $\hat{\mathbf{p}}_2$. Write down an expression for the differential cross-section in terms of the amplitude defined in (a), summed and averaged over electron spins.

Solution:

$$d\sigma=\frac{d^3p_2d^3k}{(2\pi)^6 2E_2 2k}\frac{2\pi}{2E_1 v}\delta(E_2+k-E_1)\frac{1}{2}\sum|M|^2=\frac{kdkp_2d\Omega_k d\Omega_2}{8(2\pi)^5 p_1}\frac{1}{2}\sum|M|^2$$

And $\sum|M|^2$ can be evaluated as a trace over Dirac matrices in the usual way.

c) As $|\mathbf{k}|\to 0$, the cross-section behaves like dk/k (so that the total cross-section diverges)! Identify the terms in the amplitude that give rise to this behavior and calculate the coefficient of dk/k in $d\sigma$ with $k=0$.

Solution: Looking back at the expression for $\mathcal{M}$ we see $1/k$ terms coming from the the first terms in the electron propagator expression:

$$\begin{aligned}&\left(\frac{p_2\cdot\epsilon^*}{p_2\cdot k}-\frac{p_1\cdot\epsilon^*}{p_1\cdot k}\right)ie^2\bar{u}(p_2)\gamma\cdot\tilde{A}(p_2-p_1+k)u(p_1)\sim\\ &\frac{1}{k}\left(\frac{p_2\cdot\epsilon^*}{\hat{k}\cdot\vec{p}_2-E_2}-\frac{p_1\cdot\epsilon^*}{\hat{k}\cdot\vec{p}_1-E_1}\right)ie^2\bar{u}_2\gamma\cdot\tilde{A}(p_2-p_1)u_1 \qquad (22)\end{aligned}$$

When squared this $1/k$ term leads to $kdk/k^2=dk/k$. The coefficient of dk/k in $d\sigma$ is then

$$e^4\frac{p_2d\Omega_k d\Omega_2}{8(2\pi)^5 p_1}\frac{1}{2}\left|\frac{p_2\cdot\epsilon^*}{\hat{k}\cdot\vec{p}_2-E_2}-\frac{p_1\cdot\epsilon^*}{\hat{k}\cdot\vec{p}_1-E_1}\right|^2\mathrm{Tr}(m-\gamma\cdot p_2)\gamma\cdot\tilde{A}(m-\gamma\cdot p_1)\gamma\cdot\tilde{A}^*$$

$$= e^4 \frac{p_2 d\Omega_k d\Omega_2}{4(2\pi)^5 p_1} \left| \frac{p_2 \cdot \epsilon^*}{\hat{k} \cdot \vec{p}_2 - E_2} - \frac{p_1 \cdot \epsilon^*}{\hat{k} \cdot \vec{p}_1 - E_1} \right|^2 (p_2 \cdot \tilde{A} p_1 \cdot \tilde{A}^* + p_1 \cdot \tilde{A} p_2 \cdot \tilde{A}^* - (m^2 + p_1 \cdot p_2) \tilde{A} \cdot \tilde{A}^*)$$
$$= e^2 \frac{d\Omega_k}{16\pi^3} \left| \frac{p_2 \cdot \epsilon^*}{\hat{k} \cdot \vec{p}_2 - E_2} - \frac{p_1 \cdot \epsilon^*}{\hat{k} \cdot \vec{p}_1 - E_1} \right|^2 d\sigma_{\text{elastic}} \tag{23}$$

d) Verify the gauge invariance of this leading term in the amplitude as $k \to 0$, *i.e.* that if you substitute k_μ for ϵ_μ the amplitude vanishes.

Solution: The polarization dependence of the leading term in the amplitude comes in the factor

$$\epsilon^*_\mu \left(\frac{p_2^\mu}{p_2 \cdot k} - \frac{p_1^\mu}{p_1 \cdot k} \right) \to k_\mu \left(\frac{p_2^\mu}{p_2 \cdot k} - \frac{p_1^\mu}{p_1 \cdot k} \right) = 1 - 1 = 0 \tag{24}$$

e) Suppose we try to calculate to all orders in A_{ext}. In each order, identify two graphs which will lead to a dk/k term in $d\sigma$ (do not try to show there are only two, though it is true). Show that the sum of all these graphs as $k \to 0$ is related to the amplitude for elastic scattering in the field A_{ext} and find the expression for $d\sigma$ in terms of $(d\sigma/d\Omega)_{\text{elastic}}$.

Solution: At nth order in the external potential there are $n+1$ diagrams, since the final photon can be emitted from $n+1$ different electron propagators. But the only ones which show the $1/k$ behavior are the two where the photon is emitted either before or after all the potentials act, because then the emission vertex is next to an on-shell leg $p = p_1, p_2$, so the off-shell propagator it connects to has the denominator $m^2 + (p \pm k)^2 = 2p \cdot k \to 0$ as $k \to 0$. Soft emission from any of the other $n-1$ propagators is singular when that propagator goes on-shell $q^2 + m^2$ 0, but the singularity is softened by the $\int d^4q$, provided the potential $\tilde{V}(Q)$ is less singular than $1/Q^3$ as $Q \to 0$. The expression for the sum of all these special graphs is obtained by replacing $ie\gamma \cdot \tilde{A}$ by

$$T(p_2+k, p_1) = ie\gamma \cdot \tilde{A}(p_2+k-p_1) + \int \frac{d^4q}{(2\pi)^4} ie\gamma \cdot \tilde{A}(q) S_F(p_2+k+q) ie\gamma \cdot \tilde{A}(p_2+k+q-p_1) + \cdots$$

or by

$$T(p_2, p_1-k) = ie\gamma \cdot \tilde{A}(p_2+k-p_1) + \int \frac{d^4q}{(2\pi)^4} ie\gamma \cdot \tilde{A}(q) S_F(p_2+q) ie\gamma \cdot \tilde{A}(p_2+k+q-p_1) + \cdots$$

The elastic exact scattering amplitude is $\bar{u}(p_2) T(p_2, p_1) u(p_1)$. Thus we see that the exact Bremsstrahlung amplitude has the $k \to 0$ behavior

$$\mathcal{M} \sim \left(\frac{p_2 \cdot \epsilon^*}{p_2 \cdot k} - \frac{p_1 \cdot \epsilon^*}{p_1 \cdot k} \right) e \bar{u}(p_2) T(p_2, p_1) u(p_1) = \left(\frac{p_2 \cdot \epsilon^*}{p_2 \cdot k} - \frac{p_1 \cdot \epsilon^*}{p_1 \cdot k} \right) e \mathcal{M}_{\text{elastic}}$$

f) Assume that the sum of all the other graphs not singled out in part (e) has a well-defined (*i.e.* independent of photon angles $\hat{\mathbf{k}}$) limit as $k \to 0$. Then this sum will depend on $\mathbf{p}_1$ and $\hat{\mathbf{p}}_2$ only. The amplitude coming from the graphs in (e) can be expanded in powers of k, the coefficients of each power depending on $\mathbf{p}_1$, $\hat{\mathbf{p}}_2$, $\hat{\mathbf{k}}$. Show that gauge-invariance (as stated in (d)) relates the term independent of k coming from the graphs singled out in (e) to the

$k = 0$ limit of the rest of the amplitude. Show that this means we can find the term in $d\sigma/dk$ independent of k from the graphs in (e) only.

Solution: If we replace ϵ^* by k in the amplitudes for the two classes of diagrams singled out in (e), we get

$$e\bar{u}(p_2)[T(p_2 + k, p_1) - T(p_2, p_1 - k)]u(p_1) \sim ek^\mu \bar{u}(p_2) \left[\frac{\partial T(p_2, p_1)}{\partial p_2^\mu} + \frac{\partial T(p_2, p_1)}{\partial p_1^\mu}\right] u(p_1)$$

as $k \to 0$. Call the contribution of all the other diagrams to the Bremsstrahlung amplitude $\epsilon^*_\mu(k)\mathcal{M}^\mu(p_2, k, p_1)$. Gauge invariance implies

$$k\cdot\mathcal{M} = -e\bar{u}(p_2)[T(p_2+k, p_1)-T(p_2, p_1-k)]u(p_1) \sim -ek^\mu \bar{u}(p_2) \left[\frac{\partial T(p_2, p_1)}{\partial p_2^\mu} + \frac{\partial T(p_2, p_1)}{\partial p_1^\mu}\right] u(p_1)$$

as $k \to 0$. Assuming that $\mathcal{M}^\mu$ has a well defined limit as $k \to 0$, the only way to satisfy gauge invariance is to have

$$\mathcal{M}^\mu \to -e\bar{u}(p_2) \left[\frac{\partial T(p_2, p_1)}{\partial p_2^\mu} + \frac{\partial T(p_2, p_1)}{\partial p_1^\mu}\right] u(p_1)$$

The result in part (e) seems, at first glance, to involve derivatives of the elastic amplitude w.r.t. the momenta in off-shell directions, which would require informatione not present in the elastic (on-shell) scattering amplitude. In fact, one can show that the derivatives in these off-shell directions cancel in the total amplitude. Thus not only the $O(1/k)$ **but also the** $O(k^0)$ **terms** in the bremstrahlung cross section are completely determined by the exact elastic electron scattering amplitude. This remarkable property of low energy photon emission was first discovered by F. E. Low, Physical Review **110** (1958) 974.