Standard Model/Quantun Field Theory

Problem Set 3

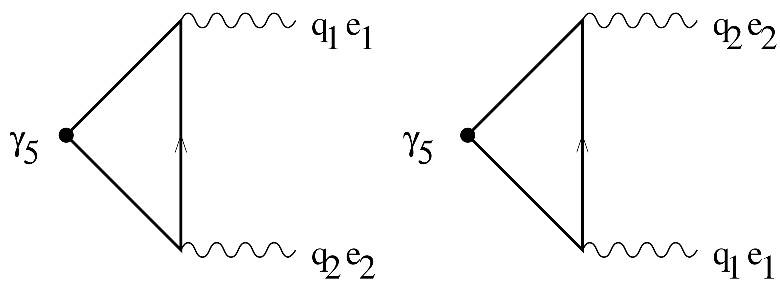
Due: 7 October 2019

Suggested reading: QFT Notes Chs 19-20; Textbook: Secs. 65; 67-68

8. As we shall see later, applying the methods for constructing a gauge invariant EM current to the axial vector current $j_5^\mu = \bar{\psi}\gamma_5\gamma^\mu\psi$ leads to the anomalous conservation law

$$\partial_\mu j_5^\mu(x) = -2m\bar{\psi}\gamma_5\psi + \frac{Q_0^2}{16\pi^2}\epsilon^{\mu\rho\nu\sigma}F_{\mu\rho}F_{\nu\sigma} \tag{1}$$

which is not zero even if $m = 0$. In this problem we shall confirm this result by explicitly calculating the matrix element $\langle 0|\bar{\psi}(x)\gamma_5\psi(x)|q_1e_1;\, q_2e_2\rangle$ of the pseudoscalar operator $\bar{\psi}(x)\gamma_5\psi(x)$ between the vacuum and a 2-photon state to lowest order in QED perturbation theory. The $e_{1,2}$ are the photon polarization vectors and $q_{1,2}$ are the photon momenta. Let $q = q_1 + q_2$. The x-dependence is of course $e^{iq\cdot x}$. You will have to evaluate the two Feynman graphs,



a) The trace calculation is not too difficult because

$$\mathrm{Tr}\gamma_5\gamma\cdot A\gamma\cdot B\gamma\cdot C\gamma\cdot D = -4i\epsilon^{\mu\nu\rho\sigma}A_\mu B_\nu C_\rho D_\sigma$$

Note the sign change from homework, which was in error! Denote this quantity by [ABCD] and remember that it is antisymmetric. All other traces of γ_5 times less than 6γs are zero. Show that after taking the traces the integrals are convergent. Use the Feynman trick

$$\frac{1}{abc} = 2\int_{x,y,z>0} \frac{dxdydz\delta(1-x-y-z)}{(ax+by+cz)^3}$$

to express the matrix elements as $[q_1q_2e_1e_2]F(q^2)$ and find $F(0)$.

Solution: With the notation

$$\mathrm{Tr}\gamma_5\gamma\cdot A\gamma\cdot B\gamma\cdot C\gamma\cdot D = -4i\epsilon^{\mu\nu\rho\sigma}A_\mu B_\nu C_\rho D_\sigma \equiv [ABCD]$$

The numerator of the first term reads

$$\begin{aligned}\text{Numerator} &= m\{[\epsilon_1(p+q_2)\epsilon_2 p] + [(p+q_1+q_2)\epsilon_1(p+q_2)\epsilon_2] + [(p+q_1+q_2)\epsilon_1\epsilon_2 p]\} \quad (2)\\ &= m\{[\epsilon_1 q_2\epsilon_2 p] + [q_1\epsilon_1(p+q_2)\epsilon_2] + [(q_1+q_2)\epsilon_1\epsilon_2 p]\} = m[q_1\epsilon_1 q_2\epsilon_2] \quad (3)\end{aligned}$$

where we exploited the antisymmetry of $[ABCD]$. Then

$$\begin{aligned}\mathcal{M} &= ie^2\int\frac{d^4p}{(2\pi)^4}\Big\{\frac{m[q_1\epsilon_1 q_2\epsilon_2]}{(m^2+(p+q_1+q_2)^2)(m^2+(p+q_2)^2)(m^2+p^2)}\\ &+(1\leftrightarrow 2)\Big\} \quad (4)\end{aligned}$$

Since the numerator is independent of p the loop integral has degree of divergence -2 and is convergent. Now we use the Feynman trick

$$\frac{1}{abc} = 2\int\limits_{x,y,z>0}\frac{dxdydz\delta(1-x-y-z)}{(ax+by+cz)^3}$$

For the first term

$$\begin{aligned}ax+by+cz &= m^2+p^2+x(q_1+q_2)^2+yq_2^2+2xp\cdot(q_1+q_2)+2yq_2\cdot p\\ &= m^2+(p+x(q_1+q_2)+yq_2)^2+xz(q_1+q_2)^2+yzq_2^2+xyq_1^2 \quad (5)\end{aligned}$$

We see that after the shift of variables $p\to p-x(q_1+q_2)-yq_2$ the two terms give identical contributions so we have after Wick rotation

$$\begin{aligned}\mathcal{M} &= -4e^2\int dxdydz\delta(1-x-y-z)\int\frac{d^4p}{(2\pi)^4}\frac{m[q_1\epsilon_1 q_2\epsilon_2]}{(m^2+p^2+xz(q_1+q_2)^2+yzq_2^2+xyq_1^2)^3} \quad (6)\\ &= \frac{me^2}{8\pi^2}\int dxdydz\delta(1-x-y-z)\frac{[q_1q_2\epsilon_1\epsilon_2]}{m^2+xz(q_1+q_2)^2+yzq_2^2+xyq_1^2} \quad (7)\\ &\to \frac{me^2}{8\pi^2}\int_0^1 dx\int_0^{1-x}dz\frac{[q_1q_2\epsilon_1\epsilon_2]}{m^2+xz(q_1+q_2)^2} \quad (8)\\ &= \frac{me^2[q_1q_2\epsilon_1\epsilon_2]}{8\pi^2q^2}\int_0^1\frac{dx}{x}\ln\frac{m^2+x(1-x)q^2}{m^2} \equiv [q_1q_2\epsilon_1\epsilon_2]F(q^2) \quad (9)\end{aligned}$$

for on-shell photons, with $q^2\equiv(q_1+q_2)^2$. We easily see that

$$F(0) = \frac{e^2}{16\pi^2 m}$$

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b) The same matrix element of the axial vector current, $J_5^\mu = \bar\psi(x)\gamma_5\gamma^\mu\psi(x)$, has a "naive divergence of $\partial_\mu J_g^\mu = (-2mi)$ times the above matrix element. The Pauli-Villars regularization of the axial current matrix element amounts to subtracting the value of the triangle graph with a large mass M from the value with the physical mass m. The r.h.s. of the divergence equation would therefore have a similar subtraction of $-2Mi$ times the result of part a) with $m\to M$. Show that this subtraction term as $M\to\infty$ gives precisely the anomalous term by comparing it to the same matrix element of the anomaly.

Solution: Putting $m = M$ and taking $M \to \infty$ gives

$$-2MiF(q^2) = \frac{-2M^2 ie^2}{8\pi^2 q^2} \int_0^1 \frac{dx}{x} \ln \frac{M^2 + x(1-x)q^2}{M^2} \to \frac{-2ie^2}{8\pi^2} \int_0^1 dx(1-x) = \frac{-ie^2}{8\pi^2}$$

So the matrix element of $-2iMj_5$ has the $M \to \infty$ limit

$$\frac{-ie^2}{8\pi^2}[q_1 q_2 \epsilon_1 \epsilon_2]$$

Now the

$$\text{Anomaly} = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \frac{e^2}{4\pi^2}\epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma$$

and the Feynman amplitude for its vacuum two photon matrix element is

$$\frac{e^2}{4\pi^2}\epsilon^{\mu\nu\rho\sigma} iq^1_\mu \epsilon^1_\nu iq^2_\rho \epsilon^2_\sigma + (1 \leftrightarrow 2) = -\frac{e^2}{2\pi^2}\epsilon^{\mu\nu\rho\sigma} q^1_\mu \epsilon^1_\nu q^2_\rho \epsilon^2_\sigma = \frac{ie^2}{8\pi^2}[q_1 q_2 \epsilon_1 \epsilon_2]$$

which is precisely the negative of the $M \to \infty$ limit of a mass M contribution to the matrix element of $(-2iM)j_5$, as desired.

9. The construction of gauge covariant currents in the nonabelian case requires a generalization of the path dependent phase $\exp[iq \int d\xi^\mu A_\mu]$ to a matrix $P[\exp[ig \int d\xi^\mu A_\mu]]$ where the P denotes path ordering analogous to time ordering.

(a) Consider a fixed curve described by $\xi^\mu(t)$, with $0 \le t \le 1$. Call the path ordered phase, for the segment of the curve from 0 to T, $W(T)$. Show that W satisfies the differential equation

$$\frac{dW}{dT} = ig\frac{d\xi^\mu}{dt}(T) A_\mu(\xi(T))W.$$

Solution: Call the path ordered phase matrix, for the segment of the curve from 0 to T,

$$W(T) = P\exp\left[ig\int_0^T dt \frac{d\xi^\mu}{dt} A_\mu\right]$$

We see that if we think of t as "time", then $W(T)$ is exactly of the form of an evolution operator for a system with "Hamiltonian" $-g\dot{\xi}^\mu A_\mu(\xi(t))$. It therefore satisfies the "Schroedinger Equation"

$$i\frac{dW}{dT} = -g\frac{d\xi^\mu}{dT} A_\mu(\xi(T))W(T)$$

as desired.

(b) For a nonabelian gauge transformation $\Omega(\xi)$, show that $W_\Omega = \Omega(\xi(T))\ W\Omega^\dagger(\xi(0))$ satisfies the equation of part (a) with gauge field

$$\Omega A_\mu \Omega^\dagger - (i/g)(\partial_\mu \Omega)\Omega^\dagger,$$

which is just the nonabelian gauge transformation of A.

Solution: For a nonabelian gauge transformation $\Omega(\xi)$, we have

$$\begin{aligned} i\frac{dW_\Omega}{dT} &= \Omega(\xi(T))i\frac{dW}{dT}\Omega^\dagger(\xi(0)) + i\frac{d\Omega(\xi(T))}{dT}W\Omega^\dagger(\xi(0)) \\ &= -g\Omega(\xi(T))\frac{d\xi^\mu}{dT}A_\mu(\xi(T))W(T)\Omega^\dagger(\xi(0)) + i\frac{d\xi^\mu}{dT}\partial_\mu\Omega(\xi(T))\Omega^\dagger(\xi(T))W_\Omega \\ &= -g\frac{d\xi^\mu}{dT}\left(\Omega A_\mu(\xi(T))\Omega^\dagger - \frac{i}{g}\partial_\mu\Omega\Omega^\dagger\right)W_\Omega(T) \end{aligned} \quad (10)$$

Thus W_Ω satisfies the equation of part (a) with gauge field

$$A_\Omega \equiv \Omega A_\mu \Omega^\dagger - (i/g)(\partial_\mu\Omega)\Omega^\dagger,$$

which is just the nonabelian gauge transformation of A. Since $W_\Omega(T)$ also satisfies the defining initial condition $W_\Omega(0)$, it follows that

$$W(A_\Omega) = \Omega(\xi(T))W(A)\Omega^\dagger(\xi(0)$$

(c) Now prove the "nonabelian" Stokes theorem for an infinitesimal closed curve $\xi(1) = \xi(0)$:

$$Pe^{ig\oint d\xi^\mu A_\mu} \approx I + ig\int d\sigma^{\mu\nu}F_{\mu\nu}, \qquad \text{for } C \text{ infinitesimal},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$.

Solution:For an infinitesimal closed curve we expand

$$\begin{aligned} P\exp\left\{ig\oint d\xi^\mu A_\mu\right\} &= I + ig\oint d\xi^\mu A_\mu + \frac{(ig)^2}{2}\oint d\xi^\mu d\eta^\nu P[A_\mu(\xi)A_\nu(\eta)] + O(L^3) \\ = I + ig&\left[\int d\sigma^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + ig\oint d\xi^\mu d\eta^\nu \theta(\xi > \eta)A_\mu(\xi)A_\nu(\eta)\right] + O(L^3) \end{aligned} \quad (11)$$

We take C so small that A_μ is constant over its extent and can be taken out of the last double integral. Then, parametrizing the curve, we note

$$\begin{aligned} \oint d\xi^\mu \oint d\eta^\nu \theta(\xi > \eta) &= \int_0^T dt \int_0^t dt' \frac{d\xi^\mu}{dt}\frac{d\xi^\nu}{dt'} = \int_0^T dt \frac{d\xi^\mu}{dt}(\xi^\nu(t) - \xi^\nu(0)) \\ &= \int_0^T dt \frac{d\xi^\mu}{dt}\xi^\nu(t) - \xi^\nu(0)(\xi^\mu(T) - \xi^\mu(0)) \\ &= \oint \xi^\nu d\xi^\mu = -\oint \xi^\mu d\xi^\nu = -2\int d\sigma^{\mu\nu} \end{aligned} \quad (12)$$

where we used $\xi(T) = \xi(0)$ because the curve is closed. Thus

$$\begin{aligned} P\exp\left\{ig\oint d\xi^\mu A_\mu\right\} &= I + ig\int d\sigma^{\mu\nu}\left[\partial_\mu A_\nu - \partial_\nu A_\mu - 2igA_\mu A_\nu\right] + O(L^3) \\ &= I + ig\int d\sigma^{\mu\nu}\left(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\right) + O(L^3) \\ &= I + ig\int d\sigma^{\mu\nu}F_{\mu\nu} + O(L^3) \end{aligned} \quad (13)$$

as desired.

To deepen your understanding of the anomaly, I strongly recommend that you use the tools developed in this exercise to establish the results quoted in the footnotes of the notes for the axial anomaly in the nonabelian case.